Entanglement and Quantum Correlations

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1 Quantum Nonlocality: EPR, Bell and GHZ

1.1 Quantum mechanics and Reality

Quantum mechanics (QM) is fundamentally different to classical mechanics. Results of measurements are not given by parameters that can considered predetermined — a superposition of two states - “the system cannot be considered to be in one state or the other until measured”. The quantum wavefunction contains inherent uncertainty- we know that, we can accept that - but the issue is more subtle than this. It is as though in QM the measurement “brings about” the result.- OK, we might say, that is OK, there is an interaction between the system and measurement apparatus, that makes sense for small systems anyway...

... TWO problems here arise

Schrodinger cat: how to interpret the QM prediction of existence of macroscopic superpositions

Einstein’s spooky action at a distance: 1935 Most important: Einstein showed there are situations where the measurements can apparently instantaneously “bring about a result” to a distant system. Now, how can the issue of interaction between measurement apparatus and system be relevant there?

These issues are are to do with ENTANGLED STATES which are at the core of the difference between QM and classical mechanics. Schrodinger introduced word ‘entangled’ states to the quantum states that seem to give this effect. Later, in Section 2 we will examine some concrete aspects of what an entangled quantum state is, within the quantum framework.

1.2 Einstein-Rosen-Podolsky (EPR) paradox

Einstein was unhappy about the assumption that quantum mechanics (QM) may be a complete theory. EPR formulated a powerful argument that quantum mechanics was incomplete in 1935. The argument is based on assumptions about the truth of “local realism” (LR), which had so far been taken for granted. In essence, the argument assumes LR, and based on the predicted existence by QM of certain entangled states, it logically argues that QM is incomplete. Thus, “completeness of QM” and LR (“locality”) are incompatible, at least if you
have entanglement. The EPR argument thus begins a journey of understanding into “quantum nonlocality”, and at the heart of this journey is the concept of entanglement.

**The EPR paradox: Bohm’s example with spins**

Look up EPR’s original paper in Physical Review A. The original argument was presented in terms of position and momentum, but we will examine Bohm’s version of the argument using spins.

Consider two spatially separated particles in the singlet spin $1/2$ state

$$|\psi\rangle = \frac{1}{\sqrt{2}} [| \uparrow_A \rangle | \downarrow_B \rangle - | \downarrow_A \rangle | \uparrow_B \rangle]$$

where $| \uparrow_k\rangle$ is the spin “up” eigenstate $|j,m\rangle$ for particle $k = A, B$, where $j = 1/2$, $m = \pm 1/2$. We can consider the three spin measurements $J_x$, $J_y$, $J_z$ that can be performed on each particle (we will denote the particle with a superscript).

**STEP 1**

There is a perfect correlation (anti-correlation) between the spins $J^A_z$ and $J^B_z$ of particle $A$ and $B$.

**Exercise 1.** Show that there is also a perfect correlation between the spin measurements $J^A_x$ and $J^B_x$, and $J^A_y$ and $J^B_y$, for this spin singlet state.

We think about correlation between spatially separated systems in the slides, by considering Bell’s famous example of Dr Bertlmann socks. This helps us understand what EPR meant when they talked about “reality” and “elements of reality”.

**STEP 2**

Assume there is no “spooky action at a distance” i.e. that locality holds. This means that the measurement at one location does not instantaneously influence the result of the spin measurement at the second location. The measurement events are to be spacelike separated to fully justify this assertion (see slides). EPR introduce premises, as stated in the slides, which these days are often referred to as “local realism.”

**STEP 3**

If Alice performs a spin $J_x$ measurement on her particle, she can predict what Bob will get if he performs a measurement $J_z$ on his particle. $LR \implies “element of reality”$ to describe Bob’s spin -ie Bob’s spin was predetermined according to parameters describing his local state- before she performs the measurement. After all, Alice cannot “steer” Bob’s state “at a distance” into one of definite spin. His spin was predetermined and since she can predict it definitely, it is predetermined to be a definite value $\lambda_x = +1/2$ OR $-1/2$.

But the same is true of spin $J_y$ ie Bob’s system has associated with it predetermined definite spins for $x$, $y$ and $z$ ie there exists three hidden variables for Bob’s measurements $\lambda_x = \pm 1/2$, $\lambda_y = \pm 1/2$ and $\lambda_z = \pm 1/2$, all of which are either $+1/2$ or $-1/2$. 


STEP 4

This is just describing classical correlation—nothing new here! ....except when you consider the quantum uncertainty relation/commutation rules for spin: which states there is NO (local) quantum state with three simultaneously (predetermined) definite spins.

STEP 5

EPR argued, based on assumed validity of local realism, that QM is incomplete!

1.3 Schrodinger’s reply—entangled states and Schrodinger’s cat

Schrodinger responded to EPR’s paper, by way of several essays. The most famous introduces the paradox of Schrodinger’s cat.

The first stage is a microscopic system in a superposition state \(|\uparrow\rangle + |\downarrow\rangle/\sqrt{2}\), which might be a spin 1/2 particle is a superposition of “spin up” or “spin down” travelling toward a Stern-gerlach apparatus, (OR it might be an alpha particle (that has decayed with 50% chance).

The next stage is a measurement device coupled to the spin eg Stern Gerlach apparatus (or Geiger counter). The coupling takes place, as an interaction \(H_{int}\), so that if \(|\uparrow\rangle\) is the initial state and that of the detector is \(|i\rangle\) then the final state of the detector and system is \(|\uparrow\rangle_{\text{needle}}|\uparrow\rangle\) which means needle pointing “up”. Similarly, if the initial state is \(|\downarrow\rangle\), the final state of the detector and system is \(|\downarrow\rangle_{\text{needle}}|\downarrow\rangle\) which means needle pointing down. Now because of the linearity of Schrodinger’s equation, if the system is initially in the superposition \(|\uparrow\rangle + |\downarrow\rangle/\sqrt{2}\), then the final state of the detector is

\[\{|\uparrow\rangle_{\text{needle}}|\uparrow\rangle + |\downarrow\rangle_{\text{needle}}|\downarrow\rangle\}/\sqrt{2}\]

which is an example of an entangled state. Schrodinger considered that the pointer needle is coupled to another system, a trigger to release poison if needle is pointing up. This poison kills cat that is located in a box, with the microscopic system. Considering the same argument as above, the final state of the system will be

\[\{|\text{dead}\rangle_{\text{cat}}|\uparrow\rangle + |\text{alive}\rangle_{\text{cat}}|\downarrow\rangle\}/\sqrt{2}\]

How do we understand the meaning of this result, in which the cat is in a superposition of alive and dead? The whole system is in a box, and an observer can look in to measure the state of the cat. But when did the cat actually die? Was it alive OR dead before the observer peered in the box? In other words, at some stage the state becomes a mixture of alive \(|\text{dead}\rangle\) and \(|\text{alive}\rangle\), and this process is called state reduction. The usual interpretation is that coupling to the environment (dissipation into a large reservoir) will cause “decay” or “decoherence” of the superposition, so it becomes a mixture. But to many this is an unsatisfactory explanation, since quantum mechanics is assumed to apply to all systems. Decoherence environments exist in principle, what happens then? Alternative theories of e.g. Diosi, Penrose, propose additions to quantum
mechanics, that will cause a state reduction for massive objects in superposition states.

1.4 Bell’s Theorem

The EPR argument was debated for some years, but a major development came with Bell’s Theorem in 1965-66. Bell’s work was important because it gave a way to directly compare the predictions of LR (via all local hidden variable theories) with the predictions of QM. He showed they were incompatible: thus QM or LR is wrong! EPR wouldn’t have thought this!

Bell was attempting to construct a theory that would include the hidden parameters λ ie the predetermined spins, and still be consistent with EPR’s no spooky action at a distance premise. He couldn’t. Here is his argument.

**STEP 1:**

**Exercise 2:** Work out the QM prediction for Bell’s hypothetical experiment.

Go back to the Bohm’s singlet state (when the spins are spatially separated, this is called the “Bell state”) and consider arbitrary spin directions: define the Pauli spin observable

\[ \sigma = \cos \theta \sigma_z + \sin \theta \sigma_x \]

Work out the prediction for the measurable expectation value of the spin product \( E(\theta, \phi) \equiv \langle \sigma^A_\theta \sigma^B_\phi \rangle \):

**STEP 2:**

What do local hidden variable theories (LHV) predict? All LHV theories put constraints on the expectation of the spin product according to Bell inequalities.

Consider first the ideal case of the singlet state which gives perfect correlation between the spin results. Recheck your calculation of Ex 1, to see that the anti-correlation is maximum (perfect) between \( \sigma^A_\theta \) and \( \sigma^B_\phi \) for all spins \( \theta \).

Then suppose EPR are right i.e. local realism is correct, and there exist hidden parameters \( \lambda^A_\theta \) to describe the spins for Bob \( (k = B) \) and for Alice \( (k = A) \). For simplicity, we can use Pauli spins, so the outcome for “spin” measurement is +1 or −1.

Then the value of \( \lambda^A_\theta \) and \( \lambda^B_\theta \) is always either +1 or −1.
Now consider the following construction for a two-setting experiment: i.e. two angles at each location

\[ B = E(\theta, \phi) - E(\theta, \phi') + E(\theta', \phi) + E(\theta', \phi') \]

**Exercise 3:**
Construct a Table of all possibilities for the LR (LHV) prediction. If spins predetermined:

\[ B = \langle \lambda_\theta \lambda_\phi \rangle - \langle \lambda_\theta \lambda_{\phi'} \rangle + \langle \lambda_{\theta'} \lambda_\phi \rangle + \langle \lambda_{\theta'} \lambda_{\phi'} \rangle \equiv (B_\lambda) \]

**Outcomes for B according to LR:**

<table>
<thead>
<tr>
<th>( \lambda_\theta )</th>
<th>( \lambda_{\theta'} )</th>
<th>( \lambda_\phi )</th>
<th>( \lambda_{\phi'} )</th>
<th>( \lambda_\theta \lambda_\phi )</th>
<th>( \lambda_\theta \lambda_{\phi'} )</th>
<th>( \lambda_{\theta'} \lambda_\phi )</th>
<th>( \lambda_{\theta'} \lambda_{\phi'} )</th>
<th>( B_\lambda )</th>
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<tr>
<td>+1</td>
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In fact, we arrive at the BELL INEQUALITY (Clauser-Horne-Shimony-Holt CHSH)

\[ |B| = |E(\theta, \phi) - E(\theta, \phi') + E(\theta', \phi) + E(\theta', \phi')| \leq 2 \]

This proof can be generalised to cases that are not perfectly correlated, so that there is a hidden variable state \( \{\lambda\} \) for each location that gives an average (probabilistic) prediction for the spin. Such a theory is a general Local Hidden Variable (LHV) theory and satisfies

\[
E(\theta, \phi) = \int_{\Lambda} \rho(\lambda) d\lambda E_A(\theta|\lambda) E_B(\phi|\lambda)
\]

\[
E(a, b) = \int_{\Lambda} \tilde{A}_a(\lambda) \tilde{B}_b(\lambda) d\rho
\]

where \( E(\theta|\lambda) \) is the expected value of the spin at Alice’s location and similarly for \( E(\phi|\lambda) \). In the second line we rewrite the equation in the notation of the Bell
proof given in the review by Clauser and Shimony, where the $a$ and $b$ are the angles, and $E_A(\theta|\lambda) = \hat{A}_a(\lambda)$ and $E_B(\phi|\lambda) = \hat{B}_b(\lambda)$. The locality assumption is that the $E(\theta|\lambda)$ does not depend on $\phi$ and $E(\phi|\lambda)$ does not depend on $\theta$, and there is the factorisation in the integrand. We will go through the proof in this case.

Proof: Taken from review Clauser and Shimony (see slides).

**STEP 3**

**Exercise 4:** check your QM prediction for the case: $\theta = 0$, $\phi = \pi/4$, $\theta' = \pi/2$, $\phi' = 3\pi/4$

The maximum violation of the Bell inequality possible algebraically is 4, but by QM is (‘Tsirelson bound’) $B = 2\sqrt{2}$.

**STEP 4**

Conclusion: QM and LHV give different predictions for a simple entangled state. Experiment needs to check which is right.

1.5 Bell experiments with photons

Most experiments on entanglement and nonlocality have most often to date dealt with photons (a recent exception violates the Bell inequality). The first photonic Bell theorem tests are from Clauser, Aspect, Zeilinger and their colleagues. We examine in the lecture experiments similar to that performed by Aspect and Zeilinger with polarised correlated photons. A source was initially two-photon atomic cascade, but later the twin output beams of the optical parametric oscillator was used.

The correlated polarised photon pair source is now used routinely as a source of “qubit” entanglement- qubit meaning two values e.g. spin up or down, in this case the two values are photon either polarised along or orthogonal to an axis.

Introduce some terminology for this.

**Quantisation of the radiation field / harmonic oscillator**

A mode of the field is quantised as a harmonic oscillator:

$$H = \hbar \omega (a^\dagger a + 1/2)$$

where $a^\dagger, a$ are creation and destruction operators $[a^\dagger, a] = 1$. The $n = a^\dagger a$ is the (photon) number operator (we sometimes drop the “hat” if meaning of operator is clear), and we can define eigenstates of this number operator $\hat{n}|n\rangle = n|n\rangle$.

The vacuum state is $|0\rangle$ and raising lowering operator rules apply: $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, $a|n\rangle = \sqrt{n}|n-1\rangle$. So, we can use symbols, $|0\rangle, |1\rangle$ to refer to spin Up or Down, OR a single or zero excitation of a mode OR whether a photon occupies polarisation mode $+$ or $-$. The most common qubit is a photon in $+$ polarised mode (bit value $+1$) versus photon in $-$ polarised mode (bit value $0$).

Common simple approach: to describe light through a beam splitter (50/50 mirror) OR polariser: creation of rotated modes

$$a_{\text{out},+} = \cos \theta a_+ + \sin \theta a_-$$

$$a_{\text{out},-} = -\sin \theta a_+ + \cos \theta a_-$$
Exercise 5: Evaluate the correlation for the output state

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |1\rangle_a + |0\rangle_a - |1\rangle_b + |0\rangle_b - |0\rangle_a - |1\rangle_a + |0\rangle_b + |1\rangle_b \right) \]

\[ = \frac{1}{\sqrt{2}} \left( a_+^\dagger b_+^\dagger - a_-^\dagger b_-^\dagger \right) |0\rangle |0\rangle |0\rangle |0\rangle \]

\[ = \frac{1}{\sqrt{2}} \left( \{ \cos \theta \hat{a}_{out,+} - \sin \theta \hat{a}_{out,-} \} \times \{ \cos \phi \hat{b}_{out,+} - \sin \phi \hat{b}_{out,-} \} \right.

\[
+ \{ \sin \theta \hat{a}_{out,+} + \cos \theta \hat{a}_{out,-} \} \times \{ \sin \phi \hat{b}_{out,+} + \cos \phi \hat{b}_{out,-} \} \right) |0\rangle |0\rangle |0\rangle |0\rangle
\]

where we have abbreviated \( |1\rangle_{out,a} + |1\rangle_{out,b} \equiv |1\rangle_{out,a} + |0\rangle_{out,a} - |1\rangle_{out,b} + |0\rangle_{out,b} - \) etc.

Evaluation of the “spin product”: probability of spin product being \(+1\) is \( \cos^2(\theta - \phi) \); probability of product being \(-1\) is \( \sin^2(\theta - \phi) \). Hence

\[ E(\theta, \phi) = \cos 2(\theta - \phi) \]

Exercise 6: Show how this result gives a violation of the CHSH Bell inequality (take the values as for the spin case dividing by 2).

This is the prediction for the photon polarisation experiments: Bell inequality is violated.

Tests of nonlocality have now expanded into other regimes, eg there are demonstrations of EPR paradox and entanglement for “CV” systems, where observables have a continuous eigenvalue spectrum. These are explained in the slides, and discussed further below. So far however, for CV systems, there has been no rigorous demonstration of a violation of a Bell inequality.
1.6 GHZ “all or nothing” multiparty nonlocality

The former Bell inequality relies on statistical collection of measurements. Greenberger-Horne-Zeilinger (GHZ) came up with a scenario in which the contradiction between QM and LR can be made for one measurement (based on previous correlations).

Consider the three “party” spin state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)$$

**STEP 1:**

Exercise 7

Now consider the product of results for spin (outcome is either $\pm 1$) if we measure two $y$ spins and one $x$ spin:

What is the QM prediction?

$$\langle \sigma_1^x \sigma_2^x \sigma_3^y \rangle = \langle \sigma_1^y \sigma_2^y \sigma_3^y \rangle = \langle \sigma_1^y \sigma_2^y \sigma_3^y \rangle = +1$$

**STEP 2**

Secret sharing:

So if Alice and Bob make their measurement of $\sigma_x$ and $\sigma_y$, they can predict that of Charlie’s $\sigma_y$. vv and etc

So, if EPR’s Local Realism (LR) is correct, each spin is described by a hidden variable eg $\lambda_x^1$ which assumes the value $+1$ or $-1$.

**STEP 3**

The hidden variables are such that $\lambda_x^1 \lambda_x^2 \lambda_y^3 = +1$

So consider the prediction by LR for the product (what is value of $(\lambda_x^1)^2$?) –always, this equals 1. So:

$$\langle \sigma_1^x \sigma_2^y \sigma_3^x \rangle = \langle \lambda_x^1 \lambda_x^2 \lambda_y^3 \rangle$$

$$= \langle \lambda_x^1 \lambda_y^2 \lambda_y^2 \lambda_y^2 \lambda_y^3 \rangle$$

$$= +1$$

**Exercise 8:** Now evaluate the quantum prediction for $\langle \sigma_1^x \sigma_2^y \sigma_3^y \rangle$? You will get $-1$! This is the opposite to the LR prediction! See Mermin’s Physics Today article.
2 Density operator formalism of entanglement

2.1 Pure states

Schrodinger introduced the term “entanglement” in his famous essay. Consider a pure \( |\psi\rangle \) for two composite systems \( A \) and \( B \) - written in terms of a basis set \( \sum_{m=-j}^{j} c_m |j,m\rangle \). The state shows entanglement between \( A \) and \( B \) iff we cannot write the state in the factorised form

\[ |\psi\rangle \neq |\psi\rangle_A |\psi\rangle_B \]

where \( |\psi\rangle_A \) is a state for Alice’s system, and similarly \( |\psi\rangle_B \) is a state for Bob’s system. For example, we cannot write the singlet \( \frac{1}{\sqrt{2}} \{ |\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \} \) in this factorised way.

2.2 Mixed states: density operator

More usually a system will be in a mixed state described by a density operator:

Recall: A density operator for a pure state \( |\psi\rangle \) is the operator \( |\psi\rangle\langle\psi| \). Note here we use bra - ket notation.

So consider spin \( \frac{1}{2} \) system.

Question 1:
Suppose the system is in the spin \( \frac{1}{2} \) "up" state: \( |\psi\rangle = |\uparrow\rangle \equiv |\frac{1}{2},\frac{1}{2}\rangle \).
What is \( \rho \)?

Remember in quantum mechanics, operators can be represented as matrices, and for this we select a suitable basis, which we call the set of states \( \{|i\rangle\} \). Then the matrix \( A \) of an operator \( O \) in that representation is given by its matrix elements \( A_{ij} = \langle i | \hat{O} | j \rangle \).

Answer: Express \( \rho \) in spinor basis \( |1\rangle \equiv |\uparrow\rangle, |2\rangle \equiv |\downarrow\rangle \), so that \( \rho_{ij} = \langle i | \rho | j \rangle \).

\[ \rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

Question 2:
What is \( \rho \) for superposition state? \( |\psi\rangle = \frac{1}{\sqrt{2}} \{ |\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \} \)

Answer:

\[ \rho = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]

Note the off-diagonal elements associated with the superposition.

Question 3:
What is the \( \rho \) for the singlet state (that we now call the Bell state).

\( |\psi\rangle = \frac{1}{\sqrt{2}} \{ |\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \} \) ?
Answer: Take a suitable basis

$$\rho_{\text{singlet}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

What is $\rho$ for the $50/50$ mixture $|\uparrow\rangle_A|\downarrow\rangle_B$ and $|\downarrow\rangle_A|\uparrow\rangle_B$?

Definition: The density operator for a mixed state (that is, a state that is in a mixture of pure states $|\psi_R\rangle$ with probability $P_R$) is given as

$$\rho = \sum_R P_R |\psi_R\rangle\langle\psi_R|$$

where $\sum_R P_R = 1$. (Prove $\text{Tr}\rho = 1$)

Question 4: What is the density matrix for the $50/50$ mixture of spin $1/2$ $|\uparrow\rangle$ and spin $-1/2$ $|\downarrow\rangle$ states?

$$\rho = \frac{1}{2} (|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)$$

Answer:

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise 9: Now evaluate density operator and matrix for the $50/50$ mixture of $|\uparrow\rangle_A|\downarrow\rangle_B$ and $|\downarrow\rangle_A|\uparrow\rangle_B$?

Now evaluate matrix $\rho$ for a spin $1/2$ system in a $50/50$ mixture of two superpositions:

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle), \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle).$$

Exercise 10: Prove for all pure states, that $\rho^2 = \rho$. Hence show that for a pure state the quantity defined as $P \equiv \text{Tr}(\rho^2)$ is $P = 1$. Also show that for a mixture, $P < 1$.

2.3 Entanglement

We say two systems $A$ and $B$ are separable iff we can express the density operator in the following factorisable form:

$$\rho = \sum_R P_R \rho^A_R \rho^B_R$$

where $\rho^A_R$ and $\rho^B_R$ are density operators for system $A$ and $B$ respectively. If this cannot be done, we say the two systems are inseparable or entangled.
Exercise 11: Take the system 50/50 mixture of singlet and triplet superpositions:

\[
\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle), \quad \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle).
\]

Is it entangled?

Exercise 12: Consider the “maximally unentangled state” for two spin 1/2 systems (two “qubits”). This is a system in an equal mixture of the 4 spin eigenstates.

\[
\rho_{\text{noisy}} = \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = \frac{1}{4}I
\]

Consider the system that is in a mixture of this maximally unentangled state \(\rho_{\text{noisy}}\) and the Bell singlet state:

\[
\rho = p\rho_{\text{singlet}} + (1-p)\rho_{\text{noisy}}
\]

where \(p\) denotes a probability (it gives the probability that the system is in \(\rho_{\text{singlet}}\).

Question 9: For what \(p\) is this state (called a Werner state) entangled?

(This is a fundamental question without an obvious immediate answer—see Peres method of Positive Partial Transpose PPT)

Please solve as follows using Peres PPT criterion (PRL, 77, 1413 1996):

We write density matrix for Werner mixed state as

\[
\rho = \begin{pmatrix}
(1-p)/4 & 0 & 0 & 0 \\
0 & (p+1)/4 & -p/2 & 0 \\
0 & -p/2 & (p+1)/4 & 0 \\
0 & 0 & 0 & (1-p)/4
\end{pmatrix}
\]

Then write the partial transpose with respect to one system only

\[
\rho^T = \begin{pmatrix}
(1-p)/4 & 0 & 0 & -p/2 \\
0 & (p+1)/4 & 0 & 0 \\
0 & 0 & (p+1)/4 & 0 \\
-p/2 & 0 & 0 & (1-p)/4
\end{pmatrix}
\]

Evaluate the eigenvalues of this new matrix.

If the system is separable (not entangled) then all eigenvalues will be nonnegative (positive partial transpose condition-PPT).

All eigenvalues are nonnegative except one which is

\[
\lambda = -(3p - 1)/4
\]

so if \(p > 1/3\) we definitely have entanglement.
For systems of dimension 2x2 (e.g. spin 1/2 by spin 1/2), the PPT criterion is necessary and sufficient for entanglement - this was proved later by the Horodecki’s (Physics Letter A223, 1, 1996; PRL78, 574 1997; 80,5239,1998). Thus there is no advantage in using the method of Wootter’s concurrence, which also gives you a necessary and sufficient condition for entanglement but only for 2x2 systems.

Hence we have separability when $p \leq 1/3$ and entanglement when $p > 1/3$.

For higher dimensional systems, it is possible to have states with a positive partial transpose that are entangled- this is called BOUND entanglement. BOUND entanglement is not distillable (see slides).

The problem of how to determine whether a $\rho$ is entangled or not is not fully solved. For spin 1/2 systems however – yes, straightforward methods exist e.g. Peres PPT and Wootter’s concurrence method.

2.4 Entanglement measures

How do we measure entanglement:

$E(\rho) \geq 0$; $E(\rho) = 0$ if state is separable; $E(\rho) = 1$ for Bell states ie those that maximally violate Bell inequalities in spin 1/2 system; is invariant under local operations and classical communication- these cannot be the source of an increased entanglement; entanglement of mixture cannot exceed the sum of the entanglement of components (convex).

**Pure state:** Entropy of entanglement $E$ measures the entanglement of a pure state and is the von Neumann entropy of the reduced density matrix defined as

$$\rho^{(A)} = \text{Tr}_B |\psi\rangle \langle \psi|$$

ie

$$E(|\psi\rangle \langle \psi|) = -\text{Tr}_{(A)} \log_2 \rho^A \equiv - \sum_i \lambda_i \log_2 \lambda_i$$

where $\lambda_i$ are eigenvalues of $\rho$ (finite dimension).

**Exercise 13:** Take the Bell singlet $|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B)$. What is the reduced density matrix for Alice’s system? What is its $E$?

$$\rho^A = \frac{1}{2} (|\uparrow\rangle \langle \uparrow| + |\downarrow\rangle \langle \downarrow|)$$

It is a 50/50 mixture. This is a maximally random mixture- only works that way for pure state because the superposition has equal amplitudes - these equal amplitudes correspond to the Bell states which show (and are the only 2 qubit states that show) a maximum violation of Bell-CHSH inequality. Thus the entropy is a measure of entanglement.

What is the $E$? (1 taking log base 2)

**Mixed states** $\rho = \sum_R P_R |\psi_R\rangle \langle \psi_R|$ are more difficult. One measure is the *Entanglement of formation*:

$$E_F = \min_{\{p,\psi_R\}} \sum_R P_R E(|\psi_R\rangle \langle \psi_R|)$$
For two qubit systems, this can be worked out, and expressed as the “concur-
currence” - Wootters. For higher dimensions the question of an entangle-
ment measure is a difficult one.

However, we will now look at some entanglement criteria: conditions that
are only satisfied if a system is entangled - these are criteria for entangle-
ment that are sufficient, but not necessary i.e. they don’t pick up all entangle-
ment, but in many cases are easy to calculate and to measure. This type of criteria
are also called “witnesses”. We will focus on criteria using uncertainty relations
and spin observables.

\subsection{Multi-partite entanglement}

It was once considered that entanglement was relevant only in a fundamental
sense - however field quantum information has emerged with a different point of
view: entanglement is a resource that can be used for applications eg quantum
cryptography, quantum teleportation ....

\textbf{Bipartite entanglement}

So, a generalised Bell state is one like

\begin{equation}
|\psi\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle
\end{equation}

or

\begin{equation}
|\psi\rangle = \cos \theta |01\rangle + \sin \theta |10\rangle
\end{equation}

How does entanglement vary with \( \theta \)? Want a simple measure of entangle-
ment. But there are four maximally entangled Bell states :

\begin{equation}
|\psi\rangle = \frac{1}{\sqrt{2}} \{ |00\rangle \pm |11\rangle \}
\end{equation}

\begin{equation}
|\psi\rangle = \frac{1}{\sqrt{2}} \{ |01\rangle \pm |10\rangle \}
\end{equation}

\textbf{Tripartite entanglement}

Have two types:

GHZ state (tend to Schrodinger cats, as number of parties become large)

\begin{equation}
|\psi\rangle = \frac{1}{\sqrt{2}} \{ |000\rangle + |111\rangle \}
\end{equation}

or \( W \)-state

\begin{equation}
|\psi\rangle = \frac{1}{\sqrt{3}} \{ |100\rangle + |010\rangle + |001\rangle \}
\end{equation}
3 Squeezing, spin squeezing and CV EPR entanglement

EPR’s original paper considered EPR correlations between the positions and momenta of two spatially separated particles. Entanglement and nonlocality can be measured with respect to observables that have continuous variable outcomes e.g. position and momentum. There have been entanglement and nonlocality experiments involving such observables, and most of these have been carried out for optical amplitudes. We revise the formalism and discuss in lectures the ways to measure such effects.

3.1 Continuous variable (cv) squeezing

Consider harmonic oscillator: in a rotating frame, we define the quadrature phase amplitudes

\[
X = a + a^\dagger \\
P = a - a^\dagger
\]

Then the uncertainty relation follows (use \([a, a^\dagger] = 1\))

\[
\Delta X \Delta P \geq 1
\]

The minimum uncertainty states are the coherent states \(|\alpha\rangle\), which are the eigenstates \(\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\), and these give \(\Delta X = \Delta P = 1\) and the “squeezed states” for which \(\Delta X = e^{-r}\), \(\Delta P = e^r\).

We have “squeezing” when \(\Delta X < 1\)

Squeezing first observed for light for (quadrature phase amplitudes) 1980’s. (LIGO and enhanced interferometry)

**How is this “phase” squeezing measured?**

Combine with large coherent field (laser) using a beam splitter (50/50 mirror) to get a measure of this fluctuation eg

\[
a_{\text{out,}+} = (a_+ + a_-)/\sqrt{2} \\
a_{\text{out,}−} = (-a_+ + a_-)/\sqrt{2}
\]

but if \(a_+\) is very large, it can be classical amplitude \(Ee^{-i\theta}\)

**Exercise 14:** Show then that the photon number difference between the two arms of the beam splitter is

\[
a_{\text{out,}+}a_{\text{out,}+}^\dagger - a_{\text{in,}+}^\dagger a_{\text{in,}+} = E(a_+ e^{i\theta} + a_- e^{-i\theta})...
\]

and that this becomes \(X\) or \(P\) depending on the choice of phase \(\theta\). We define the rotated phase quadrature amplitude as \(X_\theta = (a_+ e^{-i\theta} + a_- e^{i\theta})\).

In an experiment, we need to identify the “quantum limit”: defined as that for a coherent state, best to take vacuum \(|0\rangle\): so measure noise levels with \(a_-\) in a vacuum, then compare with noise levels when \(a_-\) is the squeezed light source.

In the optical experiments, thermal noise is insignificant.
3.2 Sources

Squeezed states are typically generated using quadratic Hamiltonians (Yuen and Caves): Take the example in the lecture.

\[ H = i\kappa E(a^2 - a^\dagger a^2) \]

Exercise 15:

Solve for the equations of motion treating the operators as time dependent (like Heisenberg equations of motion): Show that we get the following equation for \( a \). Check this - there could be a mistake!

\[ \dot{a} = \frac{-2\kappa E}{\hbar} a^\dagger \]

and the conjugate equation for \( a^\dagger \). These two give us an equation of motion for \( X = a^\dagger + a \).

\[ \dot{X} = \frac{-2E\kappa}{\hbar} X \]

and similarly (check)

\[ \dot{P} = \frac{2E\kappa}{\hbar} P \]

So, we see we have an exponential solution which gives a squeezing in the variance of \( X \) (\( X_\theta \) where \( \theta \) = ??? exercise- find the squeezing angle). Solve for the variance in the conjugate quadrature \( P \) (\( X_\theta \pm \pi/2 \)) and show that this will be increased.

3.3 Two-mode squeezing and EPR entanglement

The EPR squeezing correlation between the quadrature amplitudes of Alice and Bob for which \( X^A - X^B \) and \( P^A + P^B \) are both squeezed (there may be correlation between rotated quadratures in the intial case, but one can alter the phase of the pump \( E \) and \( \kappa \) to select which quadratures are correlated) can be obtained using the nondegenerate form of the Hamiltonian \( H = i\kappa E(ab - a^\dagger b^\dagger) \), as can be proved by simply solving the equations of motion in this case.

Note that the nondegenerate Hamiltonian

\[ H = i\kappa E(ab - a^\dagger b^\dagger) \]

can be solved similar to above (see special issue paper). Define \( X_A, P_A, X_B \) and \( P_B \) for each mode as in special issue. Solve to get?

\[ a(t) = \eta a(0) - \sqrt{(\eta^2 - 1)} b^\dagger(0) \]
\[ b(t) = \eta b(0) - \sqrt{(\eta^2 - 1)} a^\dagger(0) \]

Find the correct coefficients \( \eta \)...Find the equations and solve for the \( X_A, P_A, X_B, P_B \). Evaluate the variances in \( X_A - X_B \), \( P_A + P_B \): can they become 0 as \( \kappa \to \infty \)?

This is Einstein-Podolsky-Rosen (EPR) entanglement.
Two-mode squeezing was studied by Caves and Schumaker; the above solutions for the EPR entanglement of the two-mode squeezed state and method for detecting CV EPR correlations were shown by MR and for Parametric Oscillation (where a cavity is used) by MR and PDD. The CV EPR entanglement was demonstrated in an experiment by Ou et al in 1990’s using a cavity for enhancement of the interaction. The above technique of photon (particle) pair generation is one way for generating CV EPR entanglement. It is also possible to obtain EPR correlations by combining the outputs of two degenerate (single mode) OPO (which are modelled by the single mode quadratic Hamiltonian above) across a beam splitter. The current experiments of Schnabel create significant squeezing. Atomic homodyne has recently been realised in atom optics (see Gross et al, Nature 480, 219, 2011) where spin changing collisions are used to generate the twin atom beams where an atom pair is generated in m=1 and m=-1 hyperfine states.

3.4 CV EPR-paradox and local uncertainty relation (LUR) entanglement criteria

We can use uncertainty relations to derive criteria sufficient to deduce entanglement. We can prove a criterion for CV EPR entanglement (Tan, PRA; Duan et al, PRL): We show entanglement between the two modes if

\[ \{\Delta(X_A - X_B)\}^2 + \{\Delta(P_A + P_B)\}^2 < 4 \]

The proof is outlined below. Note this condition is not necessary and sufficient for entanglement (though a generalisation can be shown for Gaussian states (that have a positive Wigner function)).

We can also deduce entanglement by considering the EPR criterion developed by paper MR PRA 1989, based on conditional variances. We show entanglement between the two modes if

\[ \Delta(X_A|X_B)\Delta(P_A|P_B) < 1 \]

where here \(\Delta(X_A|X_B)\) is the variance of the conditional distribution \(P(X_A|X_B)\) for a result \(X_A\) given the outcome \(X_B\). The proof is outlined in ref. and will be summarised below. We note that there is asymmetry with respect to A and B in this definition. This has important consequences. We also note that in the proof of entanglement, there is here no assumption of any local uncertainty relation at the site for B. This makes the entanglement criterion “one-sided device independent”. We will see below that this EPR paradox criterion is a criterion for a special sort of entanglement that we call “EPR-steering”. We also note the simpler (thought less sensitive) alternative EPR criterion: We show EPR-steering (one-sided device-independent entanglement) between the two modes if

\[ \{\Delta(X_A - X_B)\}^2 + \{\Delta(P_A + P_B)\}^2 < 2 \]

The proof is given below.
(1) **Proof of Entanglement Criterion**

Assume separability, that the system can be described as a mixture of factorizable states, so that

\[ \rho = \sum_R P_R \rho_R^A \rho_R^B \]

where \( P_R \) is a probability (\( \sum_R P_R = 1 \)) and \( \rho_R^A \) is a quantum density operator for a state at site \( A \), and \( \rho_R^B \) one for site \( B \), etc. We follow approach of Tan and Duan et al (Physical Review Letters (PRL), 2000) and Hofman and Takeuchi (see below for reference) to derive criteria following from this assumption, that are then criteria *sufficient* (but not necessary) to demonstrate entanglement.

Assuming separability, we can write that the variance of a mixture must not be less than the average of the variances of its components. So if separability holds (no entanglement), we must have

\[
\Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) \geq \sum_R P_R(\Delta^2_R(X_A - X_B) + \Delta^2_R(P_A + P_B))
\]

\[
= \sum_R P_R[\langle X_A^2 \rangle_R + \langle X_B^2 \rangle_R - 2\langle X_A \rangle_R \langle X_B \rangle_R] + \sum_R P_R[\langle P_A^2 \rangle_R + \langle P_B^2 \rangle_R + 2\langle P_A \rangle_R \langle P_B \rangle_R] - \sum_R P_R\langle X_A - X_B \rangle^2_R - \sum_R P_R\langle P_A + P_B \rangle^2_R
\]

\[
= \sum_R P_R(\Delta^2_R X_A + \Delta^2_R P_A + \Delta^2_R X_B + \Delta^2_R P_B).
\]

Here we use that the subscript \( R \) denotes the variance or average for the state depicted by \( R \) (namely \( \rho_R^A \) or \( \rho_R^B \)). Note that separability implies the factorisation \( \langle X_A X_B \rangle_R = \langle X_A \rangle_R \langle X_B \rangle_R \), and hence the simplification. We use the result that the variance of a mixture cannot be less than the average variance of its components. For a quantum state, the following uncertainty relation follows from \( \Delta X \Delta P \geq 1 \). So the “local uncertainty relation” is

\[ \Delta^2 X + \Delta^2 P \geq 2 \]

The inequality then becomes

\[ \Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) \geq 4 \]

which gives

\[ \Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) < 4 \]

as a sufficient criterion for entanglement (note the different forms that appear in the literature depending on the choice of normalisation for the definition of \( X \) and \( P \)). Note that quantum mechanics allows the left hand side to be zero, because the commutator of \( X_A - X_B \) and \( P_A + P_B \) is zero.
(2) Proof of one-sided device-independent entanglement criterion: an EPR-steering criterion

We consider as above

$$\Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) \geq \sum_R P_R(\Delta^2_R X_A + \Delta^2_R P_A + \Delta^2_R X_B + \Delta^2_R P_B)$$

We now assume a local uncertainty relation for one site only: we assume that the site of $A$ is quantum: Hence

$$\Delta^2 X_A + \Delta^2 P_A \geq 2$$

but we do not assume quantum relations for site $B$. This implies

$$\{\Delta(X_A - X_B)\}^2 + \{\Delta(P_A + P_B)\}^2 \geq 2$$

and we observe one-sided device-independent entanglement (or EPR-steering) if

$$\{\Delta(X_A - X_B)\}^2 + \{\Delta(P_A + P_B)\}^2 < 2$$

This means we need 50% noise reduction below the quantum noise level 4.

3.5 Spin squeezing

Define the spins $J_x, J_y, J_z$: The uncertainty relation is

$$\Delta J_x \Delta J_y \geq |\langle J_z \rangle|/2$$

We say we have spin squeezing when $\Delta J_z < \sqrt{|\langle J_z \rangle|/2}$. Spin squeezing has been measured using optical Schwinger spin (polarisation modes) and more recently for cold atoms. Often, the convention is to take $(\Delta J_z)^2 < |\langle J_x \rangle|/2$, and $J_z$ is measured as the Schwinger number difference: $J_z = (a_1^\dagger a_1 - a_2^\dagger a_2)/2$, so squeezing shows as a reduced number difference fluctuation.

Note for systems of fixed spin, because of the finite dimensionality (eg for spin 1/2 the dimension is 2), there will be a limit to how much squeezing you can get. Consider spin 1/2 system. The outcomes are 1/2 or -1/2, so the maximum variance is

$$(\Delta J_y)^2 \leq 1/4$$

which means

$$\Delta J_x \geq |\langle J_z \rangle|$$

so you can’t have squeezing here (remember $\langle J_x \rangle \leq 1/2$)! More generally,

$$(\Delta J_y)^2 \leq j^2$$

so

$$\Delta J_x \geq |\langle J_z \rangle|/2j$$
More squeezing is possible with higher $j$- (ie more particles). This gives a way to deduce how many particles are in an entangled state based on the amount of spin squeezing observed (see refs.).

**Two-level atom/spin formalism**

It is useful to revise the Schwinger spin formalism that “creates” a spin system using two boson mode operators.

Let one level (e.g. of an atom) be denoted by $|1\rangle$, and the second level by $|2\rangle$.

Define spin operators according to:

$$\sigma = |0\rangle\langle 1|, \quad \sigma^\dagger = |1\rangle\langle 0|, \quad \sigma_z = (|1\rangle\langle 1| - |0\rangle\langle 0|)/2$$

**Exercise 16:** Show that these operators are indeed spin operators (need to show that they satisfy the spin commutation relations).

If we have a large number $N$ of such atoms: (levels); or two polarisation modes $\pm$ that can be occupied by large number of photons, or two levels that can be occupied by a large number of particles then it extremely useful to define Schwinger spins

$$J_z = (a_1^\dagger a_1 - a_2^\dagger a_2)/2$$
$$J_x = (a_1^\dagger a_2 + a_2^\dagger a_1)/2$$
$$J_y = (a_1^\dagger a_2 - a_2^\dagger a_1)/2i$$

so e.g. $a_1^\dagger a_1$ is the number of particles occupying level (or “state”) 1; and similarly $a_2^\dagger a_2$ is number occupying level 2. Thus $J_z$ gives “number difference” between two levels. If we have a fixed number $N$ of particles, then $a_1^\dagger a_1 + a_2^\dagger a_2$ is conserved as the total number $N$, which means $j = N/2$ is the “spin” of the system. In recent experiments, these levels are two modes of optical polarisation; two modes of two potential wells of an optical lattice, or two hyperfine atomic states.

**Exercise 17:** Check that the Schwinger spins are in fact spin operators, by evaluating the commutation relations using the boson relations $[a^\dagger, a] = 1$.

Spin squeezing was originated by Ueda and Kitagawa and can be generated where one has a nonlinearity in the Hamiltonian- eg spin squeezing has been realised (Esteve et al, Nature 464, 1165, 2010) in a two-mode BEC with a two-mode Josephson Hamiltonian $H = \chi' J_z^2 / 2 - \kappa J_x/N$ that models two weakly coupled condensates, where $\kappa$ describes a tunneling rate between them, and $\chi'$ the nonlinearity of the BEC (here $J_x$ and $J_z$ represent the Schwinger spin modes above, where $a_1$ and $a_2$ correspond to mode operators of the two condensates eg if confined to a potential well. The two mode hamiltonian might also be written $H = \kappa (a^\dagger b + ab^\dagger) + \chi a^\dagger a^2 + \chi b^\dagger b^2$. It has been shown that spin squeezing is a signature for atomic entanglement.
3.6 Spin squeezing entanglement criteria

Spin criterion 1:

Another uncertainty relation for fixed spin \( j \) is (Hofman and Takeuchi, Physical Review A, 68, 032103 (2003))

\[
(\Delta J_x^k)^2 + (\Delta J_y^k)^2 + (\Delta J_z^k)^2 \geq j
\]

and from this we can derive criteria for entanglement (Hofman and Takeuchi, PRA, 68, 032103 (2003)). Consider two systems \( A \) and \( B \): Define collective spin observables

\[
J_x = J_x^A \pm J_x^B
\]

If we have a separable state (no entanglement), then \( \rho = \sum_R P_R \rho_R^A \rho_R^B \).

Now, because the variance of a mixture can never be less than the average variance of its components, and then because for a factorised state \( \rho_R^A \rho_R^B \),

\[
\Delta(J_x^A \pm J_x^B) = \langle (J_x^A \pm J_x^B)^2 \rangle - \langle (J_x^A \pm J_x^B) \rangle^2
\]

and after using the Local Uncertainty Relation (LUR) we find that separability implies

\[
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \sum_R P_R \{(\Delta J_x)^2_R + (\Delta J_y)^2_R + (\Delta J_z)^2_R (\Delta J_x)^2_R
\]

\[
= \sum_R P_R \{(\Delta J_x)^2_R + (\Delta J_y)^2_R + (\Delta J_z)^2_R + (\Delta J_x)^2_R
\]

\[
= (\Delta J_x^A)^2 + (\Delta J_x^B)^2
\]

Thus, if

\[
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 < 2j
\]

then the two systems \( A \) and \( B \) are entangled.
Exercise: consider the Bell singlet state

$$\left| \psi \right\rangle = \frac{1}{\sqrt{2}} [\left| \uparrow \right\rangle_A | \downarrow \rangle_B - \left| \downarrow \rangle_A | \uparrow \rangle_B]$$

Will this criterion pick up the entanglement of the Bell state? What is the sum of the variances? Remember the Bohm EPR paradox, for which spins were correlated.

Answer: recall your answer to exercise 1... the EPR paradox. The spins are perfectly correlated in all directions, so the sum of the variances (with appropriate choice of a “sum” or “difference” depending on whether there is the correlation or anticorrelation) is 0.

Exercise: Does this criterion pick up the entanglement of the Werner state mixture? (see article in PRA by Hofman and Takeuchi).

Spin Criterion 2:

One can use the product uncertainty relation in a similar way: entanglement is detected if (Giovannetti et al. PRA67, 022320 (2003)).

$$(\Delta J_x)(\Delta J_y) < [||\langle J^A_x \rangle || + ||\langle J^B_x \rangle ||]/2$$

Also one can derive the following relation involving the sums of spin variances.

Assume separability (no entanglement): then, using same procedure as above,

$$\Delta^2(J_x^A \pm J_x^B) + \Delta^2(J_y^A \pm J_y^B) \geq \sum_R P_R \Delta^2_{RJ_x^A} + \Delta^2_{RJ_y^A} + \Delta^2_{RJ_x^B} + \Delta^2_{RJ_y^B}.$$ 

Now for a quantum state, the following uncertainty relation follows from $\Delta J_x \Delta J_y \geq ||\langle J_z \rangle ||/2$.

$$\Delta^2 J_x + \Delta^2 J_y \geq ||\langle J_z \rangle ||$$

(since $(x - y)^2 = x^2 + y^2 - 2xy \geq 0$). If we assume quantum state at both sites, we are able to substitute this quantum bound to derive

$$\Delta^2(J_x^A \pm J_x^B) + \Delta^2(J_y^A \pm J_y^B) \geq \sum_R P_R [\Delta^2_{RJ_x^A} + \Delta^2_{RJ_y^A} + \Delta^2_{RJ_x^B} + \Delta^2_{RJ_y^B}].$$

$$\geq \sum_R P_R [||\langle J^A_x \rangle || + ||\langle J^B_x \rangle ||]$$

Thus a criterion sufficient for entanglement is

$$\Delta^2(J_x^A \pm J_x^B) + \Delta^2(J_y^A \pm J_y^B) < ||\langle J^A_x \rangle || + ||\langle J^B_x \rangle ||$$

This criterion is like that used in “polarisation spin squeezing” experiments.
We define the Schwinger representation, so that \( J_A^x = (a_+^d a_- + a_-^d a_+)/2 \), 
\( J_B^y = i(a_+^d a_- - a_-^d a_+)/2 \), 
\( J_C^z = (a_+^d a_+ - a_-^d a_-)/2 \). 
\( N_A = a_+^d a_- + a_-^d a_+ \).

Here \( a_+ \) and \( a_- \) are mode operators for modes of eg orthogonal polarisation at each site. Equivalent relations will hold for quantum states at sites \( B \) etc, and we define similar Schwinger operators for that site. Now we examine the arrangement of Bowen et al (picture in slides), to understand one scenario in which spin squeezing is achieved. Here \( a_+ \) , \( b_+ \) are large coherent field (local oscillator) replaced by \( E \) (real) and much greater than \( a_- \) so that the Schwinger operators become 
\( N_A = N_B = E^2 \), 
\( J_A^y = J_B^y = J_C^z \), 
\( J_A^x = EX_A \), 
\( J_B^x = EX_B \). Then we get the “cv realization”, of spin squeezing when the beams \( a_- \) and \( b_- \) are EPR correlated.

### 3.6.1 Spin squeezing entanglement criterion

Consider \( N \) spin 1/2 particles (qubits) (Sorensen et al, Nature 409, 63 (2001)). Suppose there is no entanglement. Then

\[
\rho = \sum_R P_R \rho_R^1 \rho_R^2 \ldots \rho_R^N
\]

Now consider the variance of the collective spin \( J_x = \sum_{k=1}^N J_k^x \). For a separable state, this variance is constrained to be above a certain value.

Note for each spin 1/2 subsystem, there is a minimum for \((\Delta J_k^y)^2\) because there is a maximum on \((\Delta J_k^y)^2\) because the system has a finite dimension ie results \(-1/2 \) or \(1/2 \), so

\[
(\Delta J_k^y)^2 \leq 1/4
\]

Now, using the Heisenberg Uncertainty principle, \( \Delta J_x \Delta J_y \geq |\langle J_z \rangle|/2 \), so we can deduce always

\[
(\Delta J_k^y)^2 \geq |\langle J_z \rangle|^2/2
\]

for a spin 1/2 system (Sorensen and Mølmer, PRL86, 4431 (2001)). Hence, once can show for a separable state (Sorensen et al, Nature 409, 63 (2001))

\[
(\Delta J_x)^2 \geq |\langle J_x \rangle|^2/N
\]

which says that if you measure an amount of spin squeezing below a certain level,

\[
(\Delta J_x)^2 < |\langle J_x \rangle|^2/N
\]

then there must be entanglement in the system.