Adding Interactions

Jean Dalibard, *Collisional dynamics of ultra-cold atomic gases*
Proceedings of the International School of Physics "Enrico Fermi" Course CXL, Vol 140 (1999)
Important preamble:

Elastic collisions are the main and necessary tool for cooling atoms
a) Evaporative cooling -> degenerate gas
b) Sympathetic cooling -> degenerate mixture

Fermi statistics deeply affect collisions between identical fermions

Main, take-home, messages:

a degenerate gas of identical fermions is a non-interacting ideal gas
we need to “break down” the indistinguishability to access degeneracy
Low temperatures + ultralow densities

\[ \lambda_{db} \sim \mu m, \quad n^{-1/3} \sim 10^3 - 10^4 a_0 \gg \text{range of interatomic interactions } r_0 \]

\( r_0 \sim \text{van der Waals length} \sim 50 - 100 a_0 \) (\( a_0 = 0.5 \) \( \text{A} \) for alkali atoms)

Scattering never explore the short-distance details of the potential:
They can be described by a single parameter, \textbf{the scattering length} \( a \)

Binary collisions:

\textbf{two-body problem}
The scattering problem of two colliding atoms can be easily reduced to the problem of one particle with reduced mass \( M/2 \) in a molecular potential \( V(r) \).

\[
\left( -\frac{\hbar^2}{2\mu_r} \nabla^2 + V(r) \right) \psi_k(r) = E_k \psi_k(r)
\]

\( V(r) \): spherically symmetric & short-range \( r_0 \). The solution in the asymptotic limit \((r>>r_0)\) is:

\[
\psi_k(r) \propto e^{ikr} + f(k, \theta) \frac{e^{ikr}}{r}
\]

\( f(k,\theta) \): scattering amplitude

= probability amplitude to be scattered in the direction \( r/r \) under an angle \( \theta \) with respect to its initial direction \( k/k \).
Central potential \(\rightarrow\) expansion into partial waves with angular momentum \(l\)

\[
\psi_k(r) \propto e^{ikr} + f(k, \theta) \frac{e^{ikr}}{r}
\]

Ultracold: scattering at low momentum: \(k \ll 1/r_0\)

\(\rightarrow\) s-wave scattering \(l=0\) is the dominant term (\textit{if allowed by the Pauli principle})

\[
f \sim f_S = \frac{1}{2ik}(e^{2i\delta_s} - 1) = \frac{1}{kcot\delta_s - ik}
\]
Low momentum: \[ k \cot \delta_s \sim -\frac{1}{a} + r_{\text{eff}} \frac{k^2}{2} \sim -\frac{1}{a} \]

\[ a = - \lim_{k \ll 1/r_0} \frac{\tan \delta_s}{k} \]
a is the scattering length, the only parameter characterising the scattering between ultracold atoms (\(r_{\text{eff}}\) is known as the effective range of the scattering potential)

\[
\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l + 1) \sin^2(\delta_l) \quad \text{Distinguishable particles}
\]

\[
\sigma = \frac{8\pi}{k^2} \sum_{l \text{ even}} (2l + 1) \sin^2(\delta_l) \quad \text{Identical bosons}
\]

\[
\sigma = \frac{8\pi}{k^2} \sum_{l \text{ odd}} (2l + 1) \sin^2(\delta_l) \quad \text{Identical fermions}
\]

For identical fermions only odd partial waves contribute…
A simple model: scattering from a box potential

We assume a particle with momentum $k$ scattering off a box potential with a range $r_0$ and depth $V$. As before we are in the limit of low momentum scattering, $k \ll 1/r_0$. The scattering process is described by the s-wave scattering length (shift of the wave-function):

$$a = \lim_{k \ll 1/r_0} \frac{\tan \delta}{k}$$

The scattered wave-function is shifted by a distance $x$ resulting in a phase shift $\delta = kx$. Therefore, we can write:

$$a \sim \frac{\tan \delta}{k} \sim \frac{\delta}{k} = x$$

Intuitive physical interpretation of the concept of $a$:
the displacement of the wave-function by the scattering potential.

from Thomas Lompe Diploma Thesis, Heidelberg 2011
Figure 2.2: Scattering of a particle off a simple box potential. For a box with zero depth, the wavefunction of the scattered particle remains unaffected (a). For a box with a finite depth, the wavefunction is displaced by a distance $x \approx a$ by the potential (b). When the potential depth approaches the critical value $V_c$ at which a new state becomes bound, the phase shift $\delta$ of the wavefunction approaches $\pi/2$ and the scattering length diverges (c). If there is a shallow bound state in the potential, the scattering length becomes positive (d).
Bound state @ zero energy: atoms colliding at very low energies are thus in resonance with this state.

This situation corresponds to the maximum "absorption" of the incoming wave by the box potential, with an absorption cross section given by $\sigma = \frac{4\pi}{k^2}$. This is similar to the resonant absorption cross section of light by atoms, at a cross-section $\sigma_{\text{opt}} \sim \pi/k^2$ with $k$ the photon wave-vector.
\[ f(k) = \frac{1}{-\frac{1}{a} + r_{\text{eff}} \frac{k^2}{2} - ik} \quad \Rightarrow \quad \sigma = \frac{4\pi a^2}{1 + k^2 a^2} \]

1) Low momentum or small scattering length, i.e. \( k|a| \ll 1, |r_{\text{eff}}| < 1/k \):

\[ f(k) \rightarrow -a \quad \text{and the scattering cross section is given:} \quad \sigma = 4\pi a^2 \]

2) Resonant scattering, \( k|a| \gg 1, |r_{\text{eff}}| \ll 1/k \):

\[ f(k) \rightarrow \frac{i}{k} \quad \text{and the scattering cross section is given:} \quad \sigma = \frac{4\pi}{k^2} \]

The scattering is given by the square of the de Broglie wavelength of the scattering. This limit is given by quantum mechanics. In this regime \(|a| \rightarrow \infty\).

At unitarity, the scattering length diverges; the de Broglie wavelength \( 2\pi/k \) is the only relevant length scale of the scattering process.
Consequences of Fermi statistics: the strange collision properties of fermions (I)

\[ \sigma = \frac{8\pi}{k^2} \sum_{l\text{ odd}} (2l + 1) \sin^2(\delta_l) \quad \text{for identical fermions} \]

As we have seen, a pair of colliding atoms can be described by a quantum-mechanical wave-function, let’s call it \( \Psi_{\text{rel}} \).

\( \Psi_{\text{rel}} \) contains two parts – one part describing their spin \( \Xi \) and the other their relative spatial location \( \Psi_x \).

\[ \Psi_{\text{rel}} = \Xi_{\text{spin}} \Psi_x \]
Central potential: expand the spatial wave-function in terms of angular momentum partial waves, \( s, p, d, f \) \((L=0,1,2,3\hbar)\)

\[
\Psi^+_{rel} = \Xi_{spin} \Psi_x
\]

Bosons: symmetric wave-function

\[
\Psi^-_{rel} = \Xi_{spin} \Psi_x
\]

Fermions: anti-symmetric wave-function

\[
\Psi_x = \Psi_{s-wave}(L=0) + \Psi_{p-wave}(L=1) + \ldots
\]

symm. \quad \text{anti-symm.}
If the energy of the collisional energy $E$ is lower than the height of the barrier, the collision is highly suppressed.

\[
\frac{\hbar^2 l(l + 1)}{mr^2} = \text{centrifugal barrier}
\]
Identical fermions (same spin state): 

\[ \Psi_{rel}^- = \Xi_{spin} \Psi_x \]

\[ \Xi_{spin} = \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \quad \rightarrow \quad \Psi_x = \Psi_{p-wave}(L = 1) \]

\[ \frac{\hbar^2 l(l+1)}{mr^2} \quad \rightarrow \quad @ T<1 \text{ mK the collisions with } l>0 \text{ are suppressed!} \]

Collisions between identical fermions vanish @ low T
Distinguishable fermions (2 spin states):

\[ \Psi_{\text{rel}}^- = \Xi_{\text{spin}} \Psi_x \]

\[ \Xi_{\text{spin}} = \begin{array}{c}
\uparrow \\
\downarrow \\
\text{anti-symm.}
\end{array} \rightarrow \Psi_x = \Psi_{s-wave}(L = 0) \text{ symm.} \]

Collisions between distinguishable fermions DO NOT vanish @ low T
FIG. 2. Elastic cross sections vs temperature. The $s$-wave cross section ($\bigcirc$), measured using a mixture of spin states, shows little temperature dependence. However, the $p$-wave cross section ($\bullet$), measured using spin-polarized atoms, exhibits the expected threshold behavior and is seen to vary by over 2 orders of magnitude. The lines are a fit to the data, as described in the text, yielding $a_t = (157 \pm 20)a_0$. 

$\sigma = 4\pi a^2$
Consequences of Fermi statistics: *the strange collision properties of fermions (II)*

Evaporative cooling exploits elastic collisions between atoms. The initial $T$ of the atoms in the MOT is around 100 $\mu$K.

Since the only contribution to the elastic cross-section is given by the *s-wave scattering*, $\sigma = 4\pi a^2$

Efficient evaporative cooling to degeneracy is NOT possible in a sample of identical fermions.
Two COOLING strategies:

- Mixture composed by fermions in 2 spin states
- Mixture composed by fermions and bosons
  ➔ sympathetic cooling

Evaporative cooling: elastic collisions between non-identical particles
Mixture composed by fermions in 2 spin states

$^{40}$K atoms

60%-40% mixture of potassium atoms in two different spin states

Fermions are cooled to FD by means of elastic collisions with cooler gas on bosons. Evaporative cooling is performed selectively on the bosonic component: sympathetic cooling.

A. G. Truscott et al., Science 291, 2569 (2001)
Distinguishable fermions do collide \textbf{(s-wave scattering!)}

\textbf{BUT…}.

In the collisions between two Fermi gases every scattering process which produces fermions with \( k \leq k_F \) is highly inhibited! No final free momentum state is available

\textbf{Cooling fermionic systems below} \( T_F \) \textbf{is challenging}!!!
3 Pauli blocking in a Fermi gas

(a) The phenomenon of “Pauli blocking” suppresses elastic collisions in a Fermi gas. The high-energy green atom, for example, can only lose energy (green arrow) when it collides with a low-energy blue atom if it there is an unoccupied, lower-energy final state for the green atom to enter. However, all the low-energy states in the Fermi sea of atoms are already occupied, which means that this collision cannot occur. (b) Pauli blocking observed by studying “collective excitations” in a gas of potassium-40 atoms with two different spins, denoted by the Zeeman quantum numbers \( m_I = 9/2 \) and \( 7/2 \). The excitations are density oscillations, or sound waves, excited by perturbing the trapping potential. The damping time, \( T \), which is a measure of how fast the collective excitations fade away, falls below the classical value, \( T_{\text{classical}} \), as the temperature, \( T \), of the gas drops below the Fermi temperature, \( T_F \). This indicates that the collisions required to establish the collective nature of the excitation have been suppressed by Pauli blocking. (c) An absorption image of the two-component Fermi gas after it is released from the magnetic trap and allowed to expand. The relative motion in the horizontal direction of the two different types of atoms was measured by vertically separating them with a magnetic field.

The transition between a classical gas of fermions and a degenerate Fermi gas is not a phase-transition...not spectacular!!

Detecting Fermi degeneracy...a difficult task indeed

No dramatic change in the momentum distribution
Detecting Fermi degeneracy...a difficult task indeed

The mean-field interactions is almost negligible in case of identical fermions: the released energy is $E_F \gg E_{mf}$.

The momentum distribution is isotropic: no inversion in the aspect ratio.
Analysis of the momentum distribution extracted from TOF signal:
TOF signal -> momentum distribution in the trap
Classical gas -> gaussian fit
Fermi gas -> **Thomas-Fermi profile**

\[ N_{\uparrow\uparrow} = N_{\downarrow\downarrow} = 3.5 \times 10^5 \text{ @ } T/T_F < 0.1 \]
In case of thermal distribution, the width of the TOF signal is directly giving the temperature of the atoms in the trap: decreasing the width means decreased $T$!

In case of BEC, $T_C$ is determined by the wings of the thermal component!

In case of a Fermi gas, an accurate fit is needed. The size of the clouds saturates due to Fermi pressure.
PART #2

Interacting many-body systems:

a) BEC-BCS crossover: fermionic superfluidity
b) Experimental evidences: vortices and Josephson effect
We have seen that interactions between fermionic atoms at low temperatures must be s-wave scattering type (p-wave is suppressed!). In the following, we will consider the case of a Fermi gas composed of a mixture of two spin states (spin-up and spin-down).

Ex.: two internal Zeeman states of $^6\text{Li}$.
Adding interactions between fermions:

- Cooling to quantum degeneracy: *OK, we have seen it 😊😬*

- Studying superfluidity & many-body phenomena: no more pure two-body problems...

What superfluidity/superconductivity is?
Superfluidity and superconductivity are spectacular, pure, quantum phenomena:

“..the density of the helium, which at first quickly drops with the temperature, reaches a maximum at 2.2 K approximately, and if one goes down further even drops again. Such an extreme could possibly be connected with the quantum theory"

Kamerlingh Onnes, Nobel lecture 1913

Showing up in many different systems in nature:

✓ Liquid Helium: neutral atoms (1908)
✓ Superconductors: charged particles (1911)
✓ Nuclei
✓ Neutron stars
✓ Atomic Fermi gases
Examples of fermionic atoms:

- $^6\text{Li}$
- $^{40}\text{K}$
- $^{87}\text{Sr}$
- $^{171}\text{Yb}$, $^{173}\text{Yb}$
No friction & no resistance

the magnetic flux is constant along the rail

No viscosity!
Order parameter

\[ \Psi \equiv |\Psi_0| e^{i\theta} \]

- Amplitude: \(|\Psi_0|^2 = \text{superfluid density}\)
- Complex phase: \(\nabla \theta \sim \text{superfluid velocity}\)

Quantum mechanics operating at the macroscopic scale

Interactions are fundamental ingredients

&

Quantum statistics plays a relevant role
Superfluidity: condensed bosons ($T_{BEC}$) ?

Superconductor: condensed fermions ($T_F$) ?

Degenerate regime:

$$\lambda_B = \frac{h}{\sqrt{2\pi mk_B T}}$$

$$N_{\text{atoms}} = N_{\text{states}} (Q)$$

$$N_{\text{states}} = Q/\lambda^3_B$$

$T_{BEC} \approx 1/m \ (N/Q)^{2/3}$

$m = m_{\text{He}}$

$T_{BEC} \approx 3 \text{K}$

$T_F \approx 1/m \ (N/Q)^{2/3}$

$m_{\text{He}}/m_e = 7000$

$T_F \approx 50000 \text{K}$

Mechanism for pairing fermions (electrons)???

Coulomb repulsion

Critical temperature of the order of $E_F/k_B$ (thousands K), in contrast with the experimental findings: $T_c(^3\text{He}) \approx mK$
Bose-Einstein condensate

Fermi gas

SUPERFLUIDITY

SUPERFLUIDITY
What about atomic Fermi gases???
1. Typical scattering length of alkali atoms are of the order of the Van der Waals range $r_0 \approx 50-100 \ a_0$ ($a_0 =$ Bohr radius 0.5 Å).

2. Ultracold atoms are dilute: mean interparticle distances of the order of $n^{-1/3} \approx 10^4 \ a_0$.

$$k_F|a| \approx 10^{-2} \rightarrow T_c/T_F \ll 1 \ (\ll nK)$$

$\Rightarrow$ superfluidity not observable!!!

We need to increase the interactions between the atoms!!
Resonance Superfluidity in a Quantum Degenerate Fermi Gas

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We consider the superfluid phase transition that arises when a Feshbach resonance pairing occurs in a dilute Fermi gas. We apply our theory to consider a specific resonance in potassium (40K), and find that for achievable experimental conditions, the transition to a superfluid phase is possible at the high critical temperature of about 0.5TF. Observation of superfluidity in this regime would provide the opportunity to experimentally study the crossover from the superfluid phase of weakly coupled fermions to the Bose-Einstein condensation of strongly bound composite bosons.

\[
\frac{T_c}{T_F} \sim \exp\left(-\frac{\pi}{2|a|k_F}\right)
\]
The scattering processes depend on the internal structure of the particles.

- on the relative spin orientation of the valence electrons, singlet ($\uparrow \downarrow$) or triplet ($\uparrow \uparrow$).

**Example: atoms enter in the triplet channel**

**NO** coupling between $U_T$ and $U_S$

- the atoms would just scatter in $U_T$ acquiring just a phase shift (giving rise to the background scattering length $a$)

**BUT**: hyperfine interaction is NOT diagonal in the total spin $S=s_1+s_2$

→ **Coupling between $U_T$ and $U_S$**
This coupling is the essential ingredient:

FR occurs when the “incoming” state is resonant with a bound state of the singlet “closed” channel. The coupling of the open channel to the bound state in the closed channel induces a further phase shift to the scattering wave-function which changes the scattering length.

The energy difference between the incoming and the Feshbach bound state can be tuned via an applied magnetic field (different magnetic moments).

![Diagram](attachment://image.png)

**Figure 18.2** Illustration of the origin of Feshbach resonances. Atoms entering, for example, in the triplet potential are coupled to a singlet bound molecular state (potentials not to scale, sketch for illustration only). By tuning the external magnetic field, this bound state can be brought into resonance with the incoming state (at $B_0$ in the graph on the right).
The coupling induces an avoided crossing of the open channel continuum and the closed channel molecular state on resonance. This allows to transform two free atoms into a molecule and vice versa by ramping across the Feshbach resonance.
**B<B₀ (BEC_side):** The bound state is below the closed channel continuum. The scattering length is positive. The atoms can form molecules eventually condensing into a molecular Bose-Einstein condensate.

**B=B₀ (Unitary_limit):** Bound state and open channel continuum energies are degenerate and the scattering length diverges. Crossover superfluidity.

**B>B₀ (BCS_side):** The bound state is above the open channel continuum, the scattering length is negative. The atoms may form Cooper pairs and become a BCS superfluid.

\[
E_b = -\frac{\hbar^2}{ma^2}
\]
Approaching BEC-BCS crossover
There are two paradigmatic theories of superfluidity: Bose-Einstein condensation (BEC) in weakly interacting Bose gases, and Bardeen–Cooper–Schrieffer (BCS) superfluidity of long-range fermion pairs.
Bose vs Fermi superfluids

The critical temperatures for reaching the superfluid state are very different.

A weakly interacting Bose gas condenses at a temperature where the de Broglie wavelength of particles becomes comparable to the inter-particle distance, $T_c \sim \hbar^2 n^{2/3} / 2m$.

For a Fermi gas this temperature $T_F = E_F / k_B$ is set by the Fermi energy. **BUT** there is no phase transition @ $T_F$ for weakly interacting Fermi gases: the gas just forms a Fermi sea.

The critical temperature for superfluidity is given by the much lower energy scale of fermion binding. **Cooper pairing is a many-body affair**, requiring the presence of the surrounding Fermi gas.

The binding energy of Cooper pairs is much smaller than $E_F$, and the pair size is much larger than the inter-particle spacing $\sim 1/k_F$, i.e. the inverse of the Fermi wave-vector: **weak attraction regime**.
There is a regime of fermionic superfluidity that is connected to BEC: **the condensation of tightly bound fermion pairs**.

If the **attraction** between fermions **increases**, the pair size becomes much smaller than the inter-particle distance, $1/k_F$, the fermionic nature of the constituents of each pair does not longer play a role: **strong attraction regime**.

*What is between these two limiting cases of weak and strong attraction?*

In the 60s, Popov, Keldysh and Kozlov, and Eagles realised that the BCS wave function is a good description not only for long-range Cooper pair condensation, but also for a Bose–Einstein condensate of tightly bound pairs.


In 1980, Leggett used a generic two-body potential showing that the limits of tightly bound molecules and long-range Cooper pairs **are connected** in a smooth crossover:

1) The size of the fermion pairs changes from being much larger than the inter-particle spacing in the BCS limit to the small size of a molecular bound state in the BEC limit.

2) The pair binding energy varies from its small BCS value (weak, fragile pairing) to the large binding energy of a molecule in the BEC limit (stable molecular pairing).

M. Randeria, Ultracold Fermi gases: pre-pairing for condensation, Nat. Phys. 6, 561