Quantum Correlations: Entanglement

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Einstein versus Quantum Mechanics

And Snowden

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Outline

1. Non-locality and quantum mechanics
   - Einstein’s (EPR) paradox 1935
   - Schrödinger’s cat 1935 and entanglement (Leggett-Garg)
   - Bell’s theorem 1965
   - Bell and EPR experiments
   - EPR-steering
   - GHZ multipartite quantum nonlocality

2. Introduce formalism of entanglement
   - Density operator – mixed states
   - Inseparability of density matrix
   - Pauli spin examples, Werner states
   - Peres PPT criterion and concurrence
   - CV Quadrature squeezing, CV entanglement experiments
   - CV Variance entanglement and EPR criteria
   - Spin squeezing entanglement

3. Applications
   - Quantum cryptography and quantum teleportation
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Quantum mechanics and reality

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \]

- Principle of superposition

- Not one or the other until measured:
  \[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\text{dead}\rangle + |\text{alive}\rangle) \]

- Schrodinger’s cat: macroscopic superpositions
  \[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\text{dead}\rangle + |\text{alive}\rangle) \]

- Dirac

Two slits produce an interference pattern

Quantum interference of a single particle
Einstein was concerned

• “God doesn’t play dice” he said....
• He believed quantum mechanics was correct - but not complete
• There should be “hidden variables” to further describe quantum mechanics
“Shut up and calculate”
Quote David Mermin

Physicists not philosophers-
Use quantum mechanics for applications

Quantum mechanics is our most successful
And well-tested theory
And there is …Heisenberg uncertainty principle

\[ \Delta x \Delta p \geq \frac{h}{4\pi} \]

h=Planck’s constant

•You might argue to Einstein that...

•There is a fundamental indeterminacy in nature

•No quantum state can simultaneously specify both position and momentum with perfect accuracy

•If you measure position, then you disturb momentum
  • ……not so surprising..? But there is more than this
But then...Einstein presented an argument 1935

He saw how to communicate what he meant about QM:

“quantum entanglement”!

Entanglement means two particles are connected in a strange and “spooky” way (wave function can’t be factorised)
Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, Institute for Advanced Study, Princeton, New Jersey
(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.
Einstein’s entangled states show **perfect correlations** for the positions *and* momenta of two well-separated particles -

*One on earth, one on the moon*

• Alice can predict Bob’s position with **perfect accuracy**! ....

*Without disturbing his system*
Einstein-Podolsky-Rosen paradox 1935

Which means…..

Einstein versus quantum mechanics!

- Unless there is some spooky action-at-a-distance..

- Alice can measure Bob’s particle position (or momentum) without disturbing his system

- So, Bob’s particle has simultaneously a well-defined predetermined position …and momentum

- Contradicts Heisenberg uncertainty principle For a local state

- We need hidden variables !!!!
Bohm’s spin version of EPR’s argument (EPR paradox)

**Entangled states**

$$\Psi = \frac{1}{\sqrt{2}} \left( |\uparrow \downarrow\rangle - |\downarrow \uparrow\rangle \right)$$

- Spins perfectly correlated for *all* components
- Can predict any component of B, by measuring the same component of the other (A)
- EPR assume “*no spooky-action-at-a-distance*” — local realism
  then *all* spins of B are *predetermined* (“*elements of reality*”)
- Then, we need to introduce *Local Hidden Variables*
Bohm’s spin version of EPR’s argument (EPR paradox)

**Entangled states**

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- Then, we need to introduce *Local Hidden Variables*
**Einstein-Podolsky-Rosen argument (EPR paradox)**

Four steps to EPR’s argument: section 1.2 notes

**STEP 1:**
For singlet state, ALL spin components are correlated

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right) \]

Exercise 1: Show that all spin components are correlated.

\[ \sigma_\theta = \sigma_z \cos \theta + \sigma_x \sin \theta \]
Understanding EPR correlation: Dr Bertlmann’s socks

Words of John Bell to explain EPR correlation:

Dr. Bertlmann likes to wear two socks of different colours. Which colour he will have on a given foot on a given day is quite unpredictable.

But when you see (Fig. 1) that the first sock is pink you can be already sure that the second sock will not be pink. Observation of the first, and experience of Bertlmann, gives immediate information about the second. There is no accounting for tastes, but apart from that there is no mystery here. And is not the EPR business just the same?
**Bertlmann socks and correlation**

**Question:** Was the second sock “not pink” *before* the observer saw the first sock? (predetermined)

Or

Did the action of observing the first sock cause the second sock to be “not pink”?
Einstein-Podolsky-Rosen (EPR) argument: “Elements of reality”

The second sock had its colour before the observer saw the first sock.

Yes, because of past interactions, there is a correlation.

BUT The action of observing the first sock does not cause the colour of the second sock to change.

The colour of the second sock was predetermined – why?

Because we can predict the colour by measurement on another spatially separated system.

EPR call the colour of the second sock an “element of reality” of the system.

Answer: The second sock had its colour before the observer saw the first sock.
Step (2) EPR make the argument stronger: Introduce Alice and Bob

- Suppose Alice measures one sock to be **pink**
- She predicts with certainty that Bob will measure his sock to be "not pink"
- Her measurement did not cause Bob’s sock to change colour
- Einstein said, that would be like "**spooky action at a distance**"
Step (2) now look at spin and EPR’s “locality”

- Suppose Alice measures one spin to be “up”
- She knows Bob will measure his spin to be “down”
- Assume no “spooky action at a distance” – “locality”

Alice’s measurement does not change Bob’s system

- Then Bob’s spin (like the colour of the sock) is predetermined
- Bob’s spin particle has a definite value

an “Element of Reality” or “hidden variable”
EPR argument step 2: EPR are rigorous
Assume premise of local realism

EPR’s words PRA, 1935

hand, since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system.

• EPR introduce *local realism*
• Measurement by Alice doesn’t change Bob’s system “locality”
• *If the result of measurement can be predicted with absolute certainty, without disturbing the system, then that result was a predetermined property of the system—“realism”*
• Local realism implies

  *Bob’s z-spin component is predetermined (hidden variable)*
But **ALL** spin components are correlated! (step 3)

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]

- EPR’s argument: assume **local realism**
- Alice and Bob’s X- spin components are **also** perfectly correlated
- So, carry EPR argument through again- and again
- Conclude: **All** of Bob’s spin components are **completely predetermined** - hidden variables for each exist
- All his spins at any given time are either “up” or “down”
Quantum mechanics is incomplete (step 4)

EPR’s argument: assume local realism

- Conclude: All of Bob’s spin components are *completely predetermined* - hidden variables ("elements of reality") for each exist

- BUT this contradicts any local quantum description for Bob’s system Why?

- EPR conclude: *Quantum mechanics is incomplete*
EPR’s hopes of a local hidden variable (LHV) theory

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While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.
EPR argument from today’s perspective

**EPR’s argument: assumed local realism**

- Existence of EPR correlated states implies **Quantum mechanics is not complete!**
- The argument reveals the inconsistency between *premise of local realism* and *completeness of quantum mechanics*
- Later work of **BELL** showed there can be no (local realistic) completion of quantum mechanics
- Bell’s theorem indicates either **local realism** or **quantum mechanics** is wrong! -
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So - Schrodinger’s Entangled States

A (pure) entangled state is one that cannot be written in any factorised form i.e.

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle |\downarrow\rangle - |\downarrow\rangle |\uparrow\rangle)$$

$$|\phi\rangle \neq |\psi_A\rangle |\psi_B\rangle$$
What is entanglement?

Mixed states?

Entangled states are inseparable states (Werner)

Separable states satisfy:

\[ \rho = \sum_R P_R \rho_A^R \rho_B^R \]

Pure states are entangled if

\[ |\Psi\rangle_{AB} \neq |\psi\rangle_A |\psi\rangle_B \]

Separable states satisfy:

\[ P_R \quad \text{probability} \]

\[ \rho \quad \text{density operator} \]

Entangled states are inseparable states (Werner)
Separable Quantum States

Separable states are mixtures of factorised states “unentangled” states

Local density operators incorporate uncertainty principle

→ local fuzziness

Reduces correlations between A and B - can’t get EPR

$$\rho = \sum_R P_R \rho_A \rho_B$$

$P_R$ probability

$\rho$ density operator
2 Famous EPR entangled states

Entangled states are non-separable: 2 classic “EPR entanglement” states
Continuous variable (CV) or discrete outcomes

\[
|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow \downarrow \rangle - |\downarrow \uparrow \rangle)
\]

Bell state, Bohm’s EPR paradox

\[\delta(x_A - x_B)\delta(p_A + p_B)\]

• Alice can predict both Bob’s x (and p) with no fuzziness -

• Both conditional variances (of x and p) are zero:

\[
\Delta(x_B \mid x_A) \rightarrow 0 \quad \Delta(p_B \mid p_A) \rightarrow 0
\]
First Einstein entanglement experiment 1950s

Gave first evidence of Einstein Entangled states

uses polarisation of photons

C S Wu, Columbia University
(Nobel prize Awarded to Yang, Lee for theory
Wu experiment showed parity violation)
Schrodinger’s cat: quantum mechanics and a macroscopic “unreality”?

Is the moon there when nobody looks?

Yet, quantum mechanics predicts macroscopic superpositions

How does “not one or the other until measured” work for macroscopic superpositions? Do we say “dead and alive”? 

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\text{dead}\rangle + |\text{alive}\rangle)$$
Schrodinger’s cat - how is it created?

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \]

- Microscopic decay - superposition
- Interaction with measurement device that releases poison to kill cat if result is “down”
- Cat itself ends up in a superposition of dead / alive states
Schrodinger’s cat- how is it created?

Interaction of micro-system with the detector described by Hamiltonian $H_{\text{int}}$

If the initial state is $|\uparrow\rangle$ and that of detector is $|0\rangle$

then the final combined state is $|\uparrow\text{needle}\rangle|\uparrow\rangle$

If initial state is $|\downarrow\rangle$, then final state is $|\downarrow\text{needle}\rangle|\downarrow\rangle$

If initial state is the spin superposition, so that the overall initial state is

$$|\Psi\rangle_{\text{initial}} = \frac{1}{\sqrt{2}} |0\rangle(|\uparrow\rangle + |\downarrow\rangle)$$

Then the final state is (Schrodinger equation is linear)

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\text{needle}\rangle|\uparrow\rangle + |\downarrow\text{needle}\rangle|\downarrow\rangle\right)$$

Then consider the interaction with the detector and the cat, similarly, we get

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\text{alive}\rangle|\uparrow\rangle + |\text{dead}\rangle|\downarrow\rangle\right)$$
Schrodinger’s cat paradox 1935

\[ |\Psi_{\text{initial}}\rangle = \frac{1}{\sqrt{2}} |\text{alive}\rangle (|\uparrow\rangle + |\downarrow\rangle) \]

\[ |\Psi\rangle = e^{-iH_{\text{int}} t / \hbar} |\psi_{\text{initial}}\rangle = e^{-iH_{\text{int}} t / \hbar} \frac{1}{\sqrt{2}} |\text{alive}\rangle (|\uparrow\rangle + |\downarrow\rangle) \]

\[ = \frac{1}{\sqrt{2}} (|\text{alive}\rangle |\uparrow\rangle + |\text{dead}\rangle |\downarrow\rangle) \]

- Initial state is a microscopic superposition
- Interaction with the “cat” is described by a Hamiltonian \( H_{\text{int}} \)
- Time dependent Schrodinger equation \( \Rightarrow \) final state is an macroscopic superposition state

The cat is the macroscopic pointer of the measurement apparatus
Pointer “in two places at once”? Diosi/ Penrose theories propose collapse mechanism for massive objects

Diosi Penrose decoherence times for massive object m
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While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.
He proved it couldn’t be done: spin EPR state

If you do assume predetermined spins, and no “spooky action-at-a-distance” (Local Hidden Variable theory),

Then the results constrained by a Bell inequality

\[ S = E(\theta, \phi) - E(\theta', \phi) + E(\theta, \phi') + E(\theta', \phi') \leq 2 \]

YET, Quantum mechanics predicts \( S = 2\sqrt{2} \)
**BELL’S THEOREM**: let’s take a closer look

The local hidden variable (LR) prediction

---

Consider two (noncompatible) settings per site:

- Alice selects *either* $\theta$ or $\theta'$.
- Bob selects *either* $\phi$ or $\phi'$

\[ E(\theta, \phi) = \langle \sigma^A_\theta \sigma^B_\phi \rangle \]

*(note a=\(\theta\), b=\(\phi\) in diagram)*
Bell’s theorem: let’s take a closer look
The local hidden variable (LR) prediction

Recall: There is perfect correlation between Alice and Bob’s spin \( \theta \) components, and spin \( \phi \) components.

Then suppose EPR are right ie local realism is right, and there exist hidden parameters \( \lambda^k_\theta \) to describe the spins for Bob (\( k = B \)) and for Alice (\( k = A \)). For simplicity, we can use Pauli spins, so the outcome for “spin” measurement is +1 or −1.

Then the value of \( \lambda^A_\theta \) and \( \lambda^B_\phi \) is always either +1 or −1.
The **LOCAL HIDDEN VARIABLE** *(Local Realism)* prediction

Now consider the following construction for a two-setting experiment: i.e. two angles at each location

\[ B = E(\theta, \phi) - E(\theta, \phi') + E(\theta', \phi) + E(\theta', \phi') \]

**Exercise 3:**
Construct a Table of *all* possibilities for the LR (LHV) prediction. If spins predetermined:

\[ B = \langle \lambda_\theta \lambda_\phi \rangle - \langle \lambda_\theta \lambda_\phi' \rangle + \langle \lambda_\theta' \lambda_\phi \rangle + \langle \lambda_\theta' \lambda_\phi' \rangle \]
\[ = \langle \lambda_\theta \lambda_\phi - \lambda_\theta \lambda_\phi' + \lambda_\theta' \lambda_\phi + \lambda_\theta' \lambda_\phi' \rangle \equiv \langle B_\lambda \rangle \]

**Outcomes for \( B \) according to LR:**

<table>
<thead>
<tr>
<th>( \lambda_\theta )</th>
<th>( \lambda_{\theta'} )</th>
<th>( \lambda_\phi )</th>
<th>( \lambda_{\phi'} )</th>
<th>( \lambda_\theta \lambda_\phi )</th>
<th>( \lambda_\theta \lambda_{\phi'} )</th>
<th>( \lambda_{\theta'} \lambda_\phi )</th>
<th>( \lambda_{\theta'} \lambda_{\phi'} )</th>
<th>( B_\lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>+1</td>
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</tbody>
</table>
Local Hidden Variables implies Bell’s Inequality

Local hidden variables \( \Rightarrow \) Clauser-Horne-Shimony-Holt (CHSH) Bell inequality

\[ |B| = |E(\theta, \phi) - E(\theta, \phi') + E(\theta', \phi) + E(\theta', \phi')| \leq 2 \]

We assumed perfect EPR correlation, so the values of hidden variables \( \lambda \) were +1, or -1. CHSH Bell inequality still holds in presence of arbitrary correlation.
Local Hidden Variables implies Bell’s Inequality

General Bell proof by Bell from Clauser Shimony review:
We change to their notation, and note that the spin average given a hidden variable state at each site is bounded by 1

\[ E(\theta, \phi) = \int_\lambda \rho(\lambda) d\lambda E_A(\theta | \lambda) E_B(\phi | \lambda) \]

\[ E(a, b) = \int_\Lambda \bar{A}_a(\lambda) \bar{B}_b(\lambda) d\rho \]

| \bar{A}_a(\lambda) | \leq 1 \]

| \bar{B}_b(\lambda) | \leq 1. \]

(3.10)
We change notation, and note that the spin average at each site is bounded by 1,

\[ E(a, b) - E(a, b') = \int_\Lambda [\bar{A}_a(\lambda)\bar{B}_b(\lambda) - \bar{A}_a(\lambda)\bar{B}_{b'}(\lambda)] \, d\rho \]

where we take \( a' \) and \( b' \) to be alternative settings for analysers 1 and 2, respectively. This can be rewritten as:

\[ E(a, b) - E(a, b') = \int_\Lambda \bar{A}_a(\lambda)\bar{B}_b(\lambda)[1 \pm \bar{A}_{a'}(\lambda)\bar{B}_{b'}(\lambda)] \, d\rho \]

\[ - \int_\Lambda \bar{A}_a(\lambda)\bar{B}_{b'}(\lambda)[1 \pm \bar{A}_{a'}(\lambda)\bar{B}_b(\lambda)] \, d\rho. \]

Using inequalities (3.10), we then have:

\[ |E(a, b) - E(a, b')| \leq \int_\Lambda [1 \pm \bar{A}_{a'}(\lambda)\bar{B}_{b'}(\lambda)] \, d\rho + \int_\Lambda [1 \pm \bar{A}_{a'}(\lambda)\bar{B}_b(\lambda)] \, d\rho \]

or

\[ |E(a, b) - E(a, b')| \leq \pm [E(a', b') + E(a', b)] + 2 \int_\Lambda d\rho. \]

Hence:

\[ -2 \leq E(a, b) - E(a, b') + E(a', b) + E(a', b') \leq 2. \]
Quantum mechanics violates Bell inequality

\[ |B| = |E(\theta, \phi) - E(\theta, \phi') + E(\theta', \phi) + E(\theta', \phi')| \leq 2 \]

Local hidden variables \(\Rightarrow\) Clauser-Horne-Shimony-Holt (CHSH) Bell inequality

Quantum mechanics: four Bell states maximally violate CHSH Bell Inequality

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow \downarrow\rangle \pm |\downarrow \uparrow\rangle), |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow \uparrow\rangle \pm |\downarrow \downarrow\rangle) \]

\[ E(\theta, \phi) = \langle \sigma^A_\theta \sigma^B_\phi \rangle = -\cos(\phi - \theta) \quad \theta = 0, \ \theta' = \pi/2, \ \phi = \pi/4, \ \phi' = 3\pi/4 \]

\[ \Rightarrow |B| = 2\sqrt{2} \]
Exercises
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Experiments give Bell violations

Quantum mechanics gives different results to Local hidden variable theories

Experiments confirm violations of Bell inequalities - detection efficiency loopholes addressed 2013-15

$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow \downarrow\rangle - |\downarrow \uparrow\rangle)$
Experiments: Quantum mechanics OR local realism? Which one is right?

Clauser, Aspect, Zeilinger et al

Testing Bell theorem

Polarised photons
Photon pairs a,b
Polarised + or – (qubit)
Polarisation of each pair is correlated

\[
|\Psi\rangle_{source} = \frac{1}{\sqrt{2}} \left( |+\rangle_a |+\rangle_b + |−\rangle_a |−\rangle_b \right)
\]

First, we should understand the notation and predictions for this source!
Need a little formalism: harmonic oscillator

Quantisation of the radiation field / harmonic oscillator
A mode of the field is quantised as a harmonic oscillator:

\[ H = \hbar \omega (a^\dagger a + 1/2) \]

where \( a^\dagger, a \) are creation and destruction operators \([a^\dagger, a] = 1\). The \( n = a^\dagger a \) is the (photon) number operator (we sometimes drop the “hat” if meaning of operator is clear), and we can define eigenstates of this number operator \( \hat{n}|n\rangle = n|n\rangle \). The vacuum state is \(|0\rangle\) and raising lowering operator rules apply: \( a^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle \), \( a|n\rangle = \sqrt{n}|n - 1\rangle \). So, we can use symbols, \(|0\rangle, |1\rangle\) to refer to spin Up or Down, OR a single or zero excitation of a mode OR whether a photon occupies polarisation mode + or −. The most common qubit is a photon in + polarised mode (bit value +1) versus photon in − polarised mode (bit value 0).

Common simple approach: to describe light through a **beam splitter** (50/50 mirror) OR **polariser**: creation of rotated modes

\[
\begin{align*}
a_{out,+} &= \cos \theta a_+ + \sin \theta a_- \\
a_{out,-} &= -\sin \theta a_+ + \cos \theta a_- 
\end{align*}
\]

Check that the photon number conserved-
Beam splitter – polariser measurement

INPUT polarised +,- along axis
\( a_+, a_- \)

Light coming out is split into two beams:
polarised parallel and orthogonal to the polariser axis rotated by angle \( \theta \)

Consider a single photon incident: what happens?

\[
\begin{align*}
    a_{\text{out},+} &= \cos \theta a_+ + \sin \theta a_- \\
    a_{\text{out},-} &= -\sin \theta a_+ + \cos \theta a_-
\end{align*}
\]
Beam splitter – polariser measurement

The photon acts like a particle – detected at 2 or 3

Consider a single photon incident (mode $a_+$): detected at either the + or – location

Call result (Pauli spin) +1 or -1 (photon is in the superposition state)

\[
\begin{align*}
a_{out,+} &= \cos \theta a_+ + \sin \theta a_- \\
a_{out,-} &= -\sin \theta a_+ + \cos \theta a_-
\end{align*}
\]
Experiments: Quantum mechanics OR local realism? Which one is right?

Clauser, Aspect, Zeilinger et al

Polarised photons
Photon pairs a,b
Polarised + or – (qubit)
Polarisation of each pair is correlated

\[ |\Psi\rangle_{\text{source}} = \frac{1}{\sqrt{2}} \left( |+\rangle_a |+\rangle_b + |-\rangle_a |-\rangle_b \right) \]

We detect correlated photon clicks, just like the spin \( \frac{1}{2} \) particles!
Bell test – with photons and polarisers

Input to two polarising beam splitters: four modes
\[
\frac{1}{\sqrt{2}} \{ |1\rangle_a |0\rangle_b - |0\rangle_a |1\rangle_b + |0\rangle_a |0\rangle_b + |1\rangle_a |1\rangle_b \} = \frac{1}{\sqrt{2}} \{ |+\rangle_a |+\rangle_b + |-\rangle_a |-\rangle_b \}
\]
\[
= \frac{1}{\sqrt{2}} \{ |H\rangle_a |H\rangle_b + |V\rangle_a |V\rangle_b \}
\]

Correlated “qubits”

Alice rotates her polariser

Bob rotates his polariser

\[\theta\]

Source

\[\phi\]

\[a_{\text{out,}+}\]

\[b_{\text{out,}+}\]

\[a_{\text{out,-}}\]

\[b_{\text{out,-}}\]

Single detection “clicks”

Photon either + or -

Output of polarisers: calculate
\[
|\Phi\rangle = \cos(\theta - \phi) \frac{1}{\sqrt{2}} \{ |+\rangle_a |+\rangle_b + |-\rangle_a |-\rangle_b \} + \sin(\theta - \phi) \frac{1}{\sqrt{2}} \{ |+\rangle_a |-\rangle_b - |-\rangle_a |+\rangle_b \}
\]

Correlated qubits

Anti-correlated qubits
Bell test – with photons and polarisers

\[ \Phi = \cos(\theta - \phi) \frac{1}{\sqrt{2}} \{ |+\rangle_a |+\rangle_b + |-\rangle_a |-\rangle_b \} + \sin(\theta - \phi) \frac{1}{\sqrt{2}} \{ |+\rangle_a |-\rangle_b - |-\rangle_a |+\rangle_b \} \]

Correlated qubits | Anti-correlated qubits

Quantum prediction is:

\[
\Pr(+,+) = \Pr(-,-) = \cos^2(\theta - \phi); \quad \Pr(-,+)= \Pr(+,-) = \sin^2(\theta - \phi)
\]

\[
E(\theta,\phi) = \langle \sigma^A_\theta \sigma^B_\phi \rangle = \cos 2(\phi - \theta)
\]
Experiments: Quantum mechanics OR local realism?

So, which one is right?

Testing Bell theorem

Will violate the Bell inequality like before- (use same angles divided by 2!)

\[ E(\theta, \phi) = \langle \sigma^A_\theta \sigma^B_\phi \rangle = \cos 2(\phi - \theta) \]

Just one photon pair incident at a time
Alice and Bob get “click” at one of their detectors
+ or - “spin”
Experiments: Quantum mechanics OR local realism?
Which one is right? B = 2.70!

Testing Bell theorem

Clauser, Aspect, Zeilinger et al

\[ E(\theta, \phi) = \langle \sigma^A_{\theta} \sigma^B_{\phi} \rangle = \cos 2(\phi - \theta) \]

\[ \equiv \cos 2(b - a) \]

This is our “B”
Experiments: two qubit (particle) case

Testing Bell theorem

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow \downarrow\rangle - |\downarrow \uparrow\rangle) \]

Photons: all support quantum mechanics, detection inefficiencies have been problem until recently

Massive particles:
- Ions Wineland et al, excellent efficiency / poor separation
- not with space-like separated measurement events

BUT now 2015 experiments overcome loopholes
- Hensen et al (electron spin); Shalm et al, Giustina et al
Outline

1. Non-locality and quantum mechanics
   Einstein’s (EPR) paradox 1935
   Schrödinger’s cat 1935 and entanglement
   Bell’s theorem 1965
   Bell and EPR experiments
   EPR-steering
   **GHZ multipartite quantum nonlocality**

2. Introduce formalism of entanglement
   Density operator – mixed states
   Inseparability of density matrix
   Pauli spin examples, Werner states
   Peres PPT criterion and concurrence
   **CV Quadrature squeezing, CV entanglement experiments**
   CV Variance entanglement and EPR criteria
   Spin squeezing entanglement

3. Applications
   Quantum cryptography and quantum teleportation
What is your guess? Are violations of local hidden variable theories possible as the number of particles increase? How does quantum mechanics behave? Need to look at LHV versus QM predictions.

Greenberger, Horne, Zeilinger Paradoxes
See Mermin article.
Physics Today, June 9, 1990
Greenberger-Horne-Zeilinger GHZ multipartite extreme nonlocality

$$\left| \Psi \right\rangle = \frac{1}{\sqrt{2}} \left( \left| \uparrow \uparrow \uparrow \right\rangle - \left| \downarrow \downarrow \downarrow \right\rangle \right)$$

$$\langle \sigma_x^A \sigma_y^B \sigma_y^C \rangle = \langle \sigma_y^A \sigma_x^B \sigma_y^C \rangle = \langle \sigma_y^A \sigma_y^B \sigma_x^C \rangle = +1$$

What does EPR's Local realism say about this?

- Can predict any spin, by measuring other two
- Hence, LR no "spooky action at a distance" tells us each spin is predetermined
- The spins are described by hidden variables $\lambda_\theta$
  (value +1 or -1 ....so always $\lambda_\theta^2=1$, also $\lambda_x^A \lambda_y^B \lambda_y^C = +1$ etc )

$$\langle \sigma_x^A \sigma_x^B \sigma_x^C \rangle = \langle \lambda_x^A \lambda_x^B \lambda_x^C \rangle = \langle \lambda_x^A \lambda_y^B \lambda_y^C (\lambda_y^A)^2 (\lambda_y^B)^2 (\lambda_y^C)^2 \rangle = \langle \lambda_x^A \lambda_y^B \lambda_y^C \lambda_x^A \lambda_y^B \lambda_y^C \lambda_y^A \lambda_y^B \lambda_y^C \rangle = +1$$
Greenberger-Horne-Zeilinger GHZ multipartite extreme nonlocality

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle \right) \]

\[
\langle \sigma_x^A \sigma_y^B \sigma_y^C \rangle = \langle \sigma_y^A \sigma_x^B \sigma_y^C \rangle = \langle \sigma_y^A \sigma_y^B \sigma_y^C \rangle = +1
\]

What does QM say?

Mermin article, Physics Today

\[
\langle \sigma_x^A \sigma_x^B \sigma_x^C \rangle = -1
\]

The Quantum result is exactly opposite the local realism result!
Extreme violation – in one measurement!
“All or nothing” violation of LR
Greenberger-Horne-Zeilinger GHZ multipartite extreme nonlocality

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle) \]

\[ \langle \sigma^A_x \sigma^B_y \sigma^C_y \rangle = \langle \sigma^A_y \sigma^B_x \sigma^C_y \rangle = \langle \sigma^A_y \sigma^B_y \sigma^C_x \rangle = +1 \]

\[ \langle \sigma^A_x \sigma^B_x \sigma^C_x \rangle = -1 \]

Quantum secret sharing

three-photon GHZ state:

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|H\rangle_1 |H\rangle_2 |H\rangle_3 + |V\rangle_1 |V\rangle_2 |V\rangle_3) \]

where \( H \) and \( V \) denote horizontal and vertical linear polarizations

Pan et al, 2000
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Squeezing (continuous variable CV)

2.1 Continuous variable (cv) squeezing

Consider harmonic oscillator:

\[ X = a + a^\dagger \]
\[ P = (a^\dagger - a)/i \]

Correct sign here

Then the uncertainty relation follows (use \([a, a^\dagger] = 1\))

\[ \Delta X \Delta P \geq 1 \]

The minimum uncertainty states are the coherent states \(|\alpha\rangle\), which are the eigenstates \(\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\), and these give \(\Delta X = \Delta P = 1\) and the “squeezed states” for which \(\Delta X = e^{-r}, \Delta P = e^{r}\).

We have “squeezing” when \(\Delta X < 1\)

Squeezing was first observed for light for \(X\) (quadrature phase amplitudes) in the 1980’s.
Squeezing (cv)

- Amplitude squeezing
- Phase squeezing
- Squeezed vacuum
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   CV Variance entanglement and EPR criteria
   Spin squeezing entanglement (atomic entanglement)

3. Applications
   Quantum cryptography and quantum teleportation
How is squeezing generated?

Optical parametric down conversion (SPDC)
Parametric amplification (OPA)
oscillator (OPO)

Quadratic Hamiltonian

\[ H = i\kappa E(\alpha^2 - \alpha^+^2) \]

\[ X = \alpha^+ + \alpha \]

\[ P = (\alpha - \alpha^+)/i \]

For some \( \theta \):

\[ (\Delta X)^2 = e^{-rt} \]

\[ (\Delta P)^2 = e^{rt} \]

SQUEEZING!

Yuen, Caves PRA
Two-mode EPR correlations

Optical parametric down conversion (OPO)

Quadratic Hamiltonian

\[ H = i\kappa E (ab - a^+ b^+) \]

Solutions: solve for X, P as function of time, then calculate the variances for a vacuum initial state

For some \( \theta \):

\[
\begin{align*}
X_A &= a^+ + a \\
X_B &= b^+ + b \\
P_A &= (a - a^+) / i \\
P_B &= (b - b^+) / i 
\end{align*}
\]

\[
\begin{align*}
(\Delta(X_A - X_B))^2 &= (\Delta(P_A + P_B))^2 = e^{-r't} \\
(\Delta(X_A + X_B))^2 &= (\Delta(P_A - P_B))^2 = e^{r't}
\end{align*}
\]

SQUEEZING
And EPR correlations!