Information flow in open and composite quantum systems

Lecture II
Classical communication over quantum channels
Contents (Lecture II)

- Quantum states
  - Density matrices
  - Quantum channels
- Classical communication over quantum channels
  - Reminder on classical communication
  - Direct approach
  - Holevo, Schuhmacher, Westmoreland
Quantum states

- Quantum Theory is based on
  (i) Superposition principle,
  (ii) Quantum measurements

- Schroedinger dynamics in Hilbert space

• Measurement postulates:
  - Given a state (deterministic), measurement outcomes are still unpredictable. We can only predict probabilities.
  - Upon performing a measurement, the state changes “instantaneously” according to the measurement outcome (wavefunction collapse)
Doing without wave function collapse

- “There are no quantum jumps, nor are there particles” [Zeh, PLA 172, 189 (1993)]

  From a fundamental point of view, one can (and should) derive quantum measurement theory from the unitary evolution of system plus probe

- Using von Neumann measurement scheme

  - Unitary evolution in composite system, which includes Probe-1 and -2
  - JPD that ancilla 1 is in state a and ancilla 2 in state b
  - Preparation and measurement: Conditional probability
Quantum states

- For meaningful measurements, we need large samples (many repetitions)

Wave-function

Probability density

Single shot experiment

$10^3$ shots

$10^6$ shots
Quantum states

- Consider large samples – then density matrices are convenient

\[ \rho = \sum_j p_j |\psi_j\rangle \langle \psi_j| , \quad \sum_j p_j = 1 \]

- Two interpretations (underlying realities)
  - Sample of size N contains \( N \times p_j \) systems in state \( |\psi_j\rangle \)
  - Reduced state \( \rho = \text{tr}_e(|\Psi\rangle \langle \Psi|) \)

- Local measurements cannot distinguish between these cases (global measurements could)

- Properties: Hermitian, Unit trace, Positive
Quantum maps and processes

• Reduced dynamics
  \[ \Lambda_t : \varrho \rightarrow \text{tr}_e \left[ U(t) \varrho \otimes \varrho_e U(t)\dagger \right] \]

• Fluctuating Hamiltonian
  \[ \Lambda_t : \varrho \rightarrow \int u(t, x) \varrho u(t, x)\dagger \, d\mu(x) \]

• Properties
  - Linear map
  - Positive and trace-preserving (PTP)
  - But, there are PTP-maps which cannot be written as above!
    Complete positivity (CPTP)

• Representations
  - Kraus representation from the reduced dynamics
  - Supermatrix representation, – Choi representation

Bengtsson, Zyczkowski, “Geometry of quantum states” (Cambridge, 2006)
Kraus representation (from the reduced dynamics)

Quantum map:

$$\Lambda_t : \varrho_0 \rightarrow \varrho(t) = \text{tr}_e[U(t) \varrho_0 \otimes \varrho_e U(t)^\dagger] ,$$

Then with

$$\varrho_e = \sum_{\beta} \lambda_\beta \ |\beta\rangle \langle \beta| :$$

$$\varrho(t) = \sum_{\alpha\beta} \lambda_\beta \langle \alpha|U(t)|\beta\rangle \varrho_0 \langle \beta|U(t)^\dagger|\alpha\rangle = \sum_{\alpha\beta} W^{(\alpha\beta)} \varrho_0 \ W^{(\alpha\beta)^\dagger} ,$$

where it can be shown that

$$\sum_{\alpha\beta} W^{(\alpha\beta)^\dagger} W^{(\alpha\beta)} = \mathbb{1} , \quad W^{(\alpha\beta)} = \sqrt{\lambda_\beta} \langle \alpha|U(t)|\beta\rangle .$$
Kraus representation

\[ \rho = \Lambda[\rho_0] : \quad \rho = \sum_{\alpha} K_\alpha \rho_0 K_\alpha^\dagger, \quad \sum_{\alpha} K_\alpha^\dagger K_\alpha = 1 \]

- Kraus operators \( \{ K_\alpha \} \)
- Enough to know that the map is CPTP
Supermatrix representation

- Useful for algebraic operations with quantum maps (composition, inverse)
- Density matrices are converted into super-vectors by **concatenating columns**

\[
\hat{R} : \rho \quad \rightarrow \quad |\rho\rangle \\
\langle i | \rho | j \rangle \leftrightarrow \langle j, i | \rho \rangle \\
| i \rangle \langle j | \quad \mapsto \quad | j, i \rangle = | j \rangle \otimes | i \rangle
\]
Supermatrix representation

• Linear maps are converted into super-matrices

Start from quantum map and ma2svec operation $\hat{R}$:

$$\Lambda : \rho \mapsto \rho' = \Lambda[\rho], \quad \hat{R} : |i\rangle\langle j| \mapsto |j,i\rangle$$

Then

$$L = \hat{R} \circ \Lambda \circ \hat{R}^{-1}$$

$$L_{ij,kl} = \langle i, j| \hat{R} \circ \Lambda \circ \hat{R}^{-1} |k, l\rangle = \langle j| \Lambda[l]\langle k| |i\rangle$$

• Composition means matrix multiplication and the inverse is just the matrix-inverse
Choi matrix representation

\[ \Lambda \mapsto C = \sum_{i,j} |i\rangle\langle j| \otimes \Lambda[i] \langle j| \]

- Example (single qubit case)

\[ C' = \begin{pmatrix} \Lambda[0] \langle 0 | & \Lambda[0] \langle 1 | \\ \Lambda[1] \langle 0 | & \Lambda[1] \langle 1 | \end{pmatrix} \]

- Properties:
  - Hermitian
  - Diagonal blocks are density matrices
  - If and only if the map is CPTP, then its Choi representation is positive (Choi theorem)

Classical communication over a quantum channel

- Wrapping a classical channel around a quantum one
  - Simple classical capacity
  - Direct (naive) concept
  - HSW concept

Classical capacity

\[ H(X) : \text{Information unknown to Bob} \]
\[ H(X, Y) - H(Y) : \text{Info. unknown after one channel use} \]

\text{Info. Gain per one channel use:}
\[ H(X:Y) = H(X) - H(X, Y) + H(Y) \]
Binary symmetric channel

- Alice sends 01001...
- Bob receives 01011...

- Joint probabilities

<table>
<thead>
<tr>
<th>Bob\Alice</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1−p</td>
<td>p</td>
</tr>
<tr>
<td>1</td>
<td>p</td>
<td>1−p</td>
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</tbody>
</table>

- Alice: \( \vec{p}_x = \begin{pmatrix} q \\ 1 − q \end{pmatrix} \)

- Bob: \( \vec{p}_y = \begin{pmatrix} q + p - 2pq \\ 1 + 2pq - q - p \end{pmatrix} \)

- Joint probabilities

\[
p(k, j) = \begin{pmatrix} (1 − p)q & p(1 − q) \\ pq & (1 − p)(1 − q) \end{pmatrix}
\]
Capacity of the binary symmetric channel

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\[ \tilde{p}_x = \begin{pmatrix} q \\ 1 - q \end{pmatrix} \]

Capacity in units of yes/no questions.

Info received saves Bob from so many yes/no questions.

Noisy channel coding theorem

\[ C = \max_q H(X : Y) \]
Capacity of the binary asymmetric channel

\begin{align*}
\begin{array}{c|cc}
\text{Bob} & 0 & 1 \\
\hline
0 & 1 - p_1 & p_2 \\
1 & p_1 & 1 - p_2 \\
\end{array}
\end{align*}

C = \max_q H(X : Y)
Classical communication over a quantum channel

- Wrapping a classical channel around a quantum one

$$\{ \varrho_j \} \rightarrow \{ \varrho'_j = \Lambda[\varrho_j] \} \rightarrow \{ E_k \}$$

$$\varrho_\alpha \rightarrow E_\alpha$$

- As before, Alice chooses $\varrho_j$ with probabilities $p_x(j)$.

- Bob applies the POVM $\{ E_k \}$ and hence obtains results $E_k$ with probabilities

$$p_y(k|j) = \text{tr} \left( E_k \Lambda[\varrho_j] \right)$$

- We have all ingredients to compute classical capacity

$$C = \max_{p_x(j), \varrho_j} \quad H(X : Y)$$

[Fuchs, PRL 79, 1162 (1997); Kholevo, Probl. Inf. Transm. 9, 177 (1973)]
• Theorem

\[ C^{(1)} = \max_{p_x(j), \varrho_j} \left[ S\left( \sum_j p_j \varrho_j' \right) - \sum_j p_j S(\varrho_j') \right], \quad \varrho_j' = \Lambda[\varrho_j] \]

\[ S(\varrho) = - \sum_j \lambda_j \log_2(\lambda_j), \quad \varrho = \sum_j \lambda_j |j\rangle\langle j| \]

• Holevo bound for mutual information

\[ H(X : Y) \leq S\left( \sum_j p_x(j) \varrho_j \right) - \sum_j p_x(j) S(\varrho_j) \]

• States prepared by Alice must be independent, but Bob is allowed to make global measurements on several subsequent states, received.

• Fuchs: C can be strictly smaller; non-orthogonal inputs may be optimal but only to \( C^{(1)} \) not to C

\[ C' \leq C^{(1)} \]
End