Lecture 2

How to characterize or quantify 2-mode correlations.

correlation functions
different levels of correlation (gemelity, etc.)
Cauchy-Schwarz violations
EPR correlations

quantum correlations

An experimentalist view of the world: We spend much of our time looking for interesting differences between classical and quantum mechanics so that we can measure them (and write them up and publish them). Occasionally these differences also turn out to be tremendously useful and can generate applications.

A large number of different measures of correlation as well as of quantum entanglement exist, and it is useful to know and understand the differences between them. I will select a few of the many possible measures and discuss them in the following.

non-separable states

A fundamental difference between “classical” states of light (which don’t actually exist) and quantum states is the ability of quantum states to have stronger correlations than allowed for classically. Even if the system can be easily separated into two parts, the state cannot be written as a tensor product of states of the two sub-systems.

\[ |\Psi\rangle \neq |\Psi_1\rangle \otimes |\Psi_2\rangle \]

Strong correlations exist between the subsystems in such cases. The source of these correlations is not mysterious; it is due to an earlier interaction of the two subsystems!
4-wave mixing and twin beams

Here a local nonlinear optical interaction generates two distinct beams of photons, travelling in different directions which become the two sub-systems that we will consider.
classical correlations

We do, however, need to be careful about what sort of correlations we are discussing.

I can throw balls (or photons) in two directions and, as long as I am careful, I can keep the number of balls thrown in each direction the same.

This is a strong correlation in the number of balls, but it is not particularly special or non-classical – there must be more to it!
Consider two modes of the field that will make up our subsystems. These beams can have different frequencies, directions, polarizations, spatial extent or shape. They have creation and annihilation operators $\hat{a}$, $\hat{b}$, $\hat{a}^\dagger$, and $\hat{b}^\dagger$. 

We can write quadrature operators for the fields 
$\hat{X}^+ = \hat{a} + \hat{a}^\dagger$ and $\hat{X}^- = -i(\hat{a} - \hat{a}^\dagger)$
fluctuations and correlations

You can measure intensity, or by homodyne measurement, a quadrature of the field.

The fluctuations, (Fano factor) $F = \langle \delta X^2 \rangle$ represents the noise of the field quadrature, normalized to shot noise.

If we measure the fluctuations in the two beams we can write down a correlation coefficient

$$C_{12} = \frac{\langle \delta X_1 \delta X_2 \rangle}{\left( \langle \delta X_1^2 \rangle \langle \delta X_2^2 \rangle \right)^{1/2}}$$
correlation coefficient

$$C_{12} = \frac{\langle \delta X_1 \delta X_2 \rangle}{\left(\langle \delta X_1^2 \rangle \langle \delta X_2^2 \rangle \right)^{1/2}}$$

$C_{12} = 1$ for perfect correlation and $C_{12} = -1$ for perfect anti-correlation; the field takes both positive and negative values.

Assume positive correlations.

When are the correlations “quantum” or “non-classical” in nature?
Intensity correlations or fluctuations can be used to identify non-classical states.

A first level of “quantumness” is that the field cannot be described by a classical field with classical fluctuations; that is, it cannot be generated just by radiating classical electron currents.

Photon anti-bunching in resonance fluorescence was the first non-classical field state observed experimentally.

The two-time correlation function must always decrease from $t=0$, and this is violated in cases of photon antibunching.
The two-time correlation function must always decrease from \( t=0 \), and this is violated in cases of photon antibunching.

these are non-classical correlations!

figures, R. Short, thesis, U. Rochester
photon antibunching

The Cauchy-Schwartz inequality applies to all classical fields:
\[ <XY>^2 \leq <X^2><Y^2> \]

\[ C_{12}(\tau) = \frac{<I_1(t)I_2(t+\tau)>}{(<I_1><I_2>)} \]

since \[ <X>^2 \leq <X^2> \] (Cauchy-Schwarz for \( Y=1 \))
then \[ C_{11}(0) = \frac{<I_1^2>}{<I_1>^2} \geq 1 \]

\[ C_{11}(\tau) = \frac{<I_1(t)I_1(t+\tau)>}{(<I_1><I_1>)} \rightarrow \frac{<I_1(t)>}{<I_1(t+\tau)>}/(<I_1><I_1>) \]
as the fields will be uncorrelated as \( \tau \rightarrow \infty \)
and \[ C_{11}(\tau \rightarrow \infty) = 1 \]

\[ C_{12} \leq (C_{11}C_{22})^{1/2} \] (divide C-S by \( <I_1><I_2>^2 \) and take the square root)
more C-S relations

The Cauchy-Schwartz inequality:
\[ <XY>^2 \leq <X^2> <Y^2> \]

\[
C_{12}(\tau)^2 = \frac{<I_1(t)I_2(t+\tau)>^2}{(<I_1><I_2>)^2} \\
\leq \frac{<I_1(t)^2><I_2(t+\tau)^2>}{(<I_1><I_2>)^2} \\
\leq \frac{<I_1(0)^2><I_2(0)^2>}{(<I_1><I_2>)^2} \\
\leq C_{11}(0)C_{22}(0)
\]

for \( I_1 = I_2 \) this implies that \( C_{11}(\tau) \leq C_{11}(0) \)  
(for a classical field!)

\[
C_{12}(0) \leq (C_{11}(0)C_{22}(0))^{1/2} \quad (\text{divide C-S by } (<I_1><I_2>)^2 \text{ and take sqrt})
\]

Note that for two fields (rather than intensities) the correlations can run between +1 and -1.
Intensity correlations ($g^{(2)}(\tau)$)

Cauchy-Schwarz inequality: a measure of non-classicality

\[
\left[ g_{pc}^{(2)}(\tau) \right]^2 \leq g_{pp}^{(2)}(0) g_{cc}^{(2)}(0) \quad \text{or} \quad <I_1(t)I_2(t+\tau)^2> \leq <I_1^2(t)> <I_2^2(t)>
\]

where

\[
g_{ab}^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{b}^\dagger(t+\tau)\hat{b}(t+\tau)\hat{a}(t) \rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle \langle \hat{b}^\dagger(t)\hat{b}(t) \rangle}
\]

Notice:
- the detector efficiency doesn’t matter!
- for a fixed level of correlations between the beams, one way to make the violations large is to make the intensities of the beams small!
Cauchy-Schwarz violations

Note that the “non-classical” property is all that one can determine in this way; if it violates Cauchy-Schwarz, it is non-classical.

It does not matter how much it violates this inequality by – either a large or small amount, or a few or many standard deviations – a C-S violation can only say that the light is non-classical, and “large violations” say nothing about the field being “more quantum” than for small violations!
Intensity correlations \( (g^{(2)}(\tau)) \)

\[
\left[ g_{pc}^{(2)}(\tau) \right]^2 \leq g_{pp}^{(2)}(0) g_{cc}^{(2)}(0)
\]

where

\[
g_{ab}^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t) \hat{b}^\dagger(t + \tau) \hat{b}(t + \tau) \hat{a}(t) \rangle}{\langle \hat{a}^\dagger(t) \hat{a}(t) \rangle \langle \hat{b}^\dagger(t) \hat{b}(t) \rangle}
\]

We measure the correlation function of the fluctuations in the form

\[
g_{ab}^{(2)}(\tau) = 1 + \varepsilon_{ab}
\]

(our beams are bright and \( g^{(2)} \) only deviates from 1 by a small amount)

Cauchy-Schwartz inequality takes the form (lowest order in \( \varepsilon \))

\[
\varepsilon_{pc} \leq \frac{\varepsilon_{pp} + \varepsilon_{cc}}{2}
\]
Cauchy-Schwarz violations with squeezed light

\[ \varepsilon_{pc} \leq \frac{\varepsilon_{pp} + \varepsilon_{cc}}{2} \]

>1 MHz sharp high-pass filter

~5 MHz detector roll-off
Cauchy-Schwartz vs. Squeezing

filtering of the signal matters
- if the squeezing spectrum is such that you only have squeezing over a certain frequency range, then the violation of the C-S inequality will be maximized if you filter the signal to this same frequency range
Cauchy-Schwarz violations 2

\[ \frac{\varepsilon_{pp} + \varepsilon_{cc}}{2} \]

1-2 MHz sharp filters
twin beams

Not just optical fields, but other physical systems can display “twin beam” properties too...
- look at continuous variable properties of fields

-> matter waves – coherent and with a phase
  ⇒ We can do homodyne detection with matter waves as well
  ⇒ We can also transfer photon correlations / entanglement to atoms

Non-Classical states can be detected by
- Intensity correlations between 2 photocurrents
- Intensity fluctuations of one photocurrent

⇒ this is a low-level or qualitative statement of “quantumness”
- photon counting statistics ~ “digital”
- continuous variables ~ intensities or field magnitudes
twin beams (cont.)

We will work with intense or bright beams or at least continuous variables.

- The “classical limit” or “standard quantum limit” (we are stuck with quantum mechanics, after all - we can’t really turn off Q. M. and look at a classical world.)

In the closest thing we have to a classical world the fluctuations of a field

$$F_i = \langle (X-\langle X \rangle)^2 \rangle = \langle \delta X_i^2 \rangle = 1$$

the Fano factor for the field (normalized to the SQL)

This is the case for coherent states and the special case of vacuum states of the field. In these cases: $F = 1$. 
twin beams (cont.)

Linear or passive optics preserve the classical character of a field - these are energy-preserving elements.

With a non-classical field such elements can either preserve the non-classical nature, or “degrade” it – that is, push it toward the classical.

- beamsplitters, mirrors, free propagation....

It requires non-linear elements to create non-classical fields.
correlations in twin beams

look at “twin beams” by examining their correlations

50/50 beamsplitter

F\text{in} \quad \rightarrow \quad |0\rangle \quad \rightarrow \quad F

\begin{align*}
|C_{12}|_{\text{classical}} &= \frac{F_{\text{in}} - 1}{F_{\text{in}} + 1} = 1 - \frac{1}{F}
\end{align*}

Note that for large input classical fluctuations (\(F_{\text{in}} \rightarrow \infty\)) one has \(|C_{12}| \rightarrow 1\).
classical correlations

⇒ Large correlations do not imply anything “quantum” if there are large classical fluctuations, then coupling-in the vacuum field doesn’t matter much...

\[ |0\rangle \rightarrow \frac{1}{\sqrt{2}} F \]

\[ F_{\text{in}} \]

\[ |C_{12}| \rightarrow 1 \]
Gemelity

or “twinness” (?)

A measure of twin-beam correlations

What does it take to verify that there are non-classical or quantum correlations?

\[ \delta X_{\text{out}} = r\delta X_1 - t\delta X_2 \]

adjust the reflection/transmission of beamsplitter to minimize the noise

- If you can adjust the reflectivity to make the noise on the output beam less than shot noise for the intensity, then the correlations between $X_1$ and $X_2$ are non-classical. This can be quantified by the Gemelity, $G$.

$$G = \frac{F_1 + F_2}{2} - \sqrt{C_{12} F_1 F_2 + \left(\frac{F_1 - F_2}{2}\right)^2}$$

$G < 1$ implies that there are non-classical correlations between $X_1$ and $X_2$
gemelity...

- You won’t actually see this measure used very often.
- It is a minimal measure of non-classical correlations
- It is somewhat difficult to implement in the lab.

eexample: 2 beams with equal mean intensities and noise

\[ F_1 = F_2 = F = \langle \delta X_i^2 \rangle \text{ and } G = F(1-|C_{12}|) \]

\[ \delta X_{\text{out}} = r\delta X_1 - t\delta X_2 = \frac{1}{\sqrt{2}} (\delta X_1 - \delta X_2) \]

and

\[ G = \frac{\langle (\delta X_1 - \delta X_2)^2 \rangle}{2} \]

This is the normalized difference between the fluctuations of the two beams.
balanced classical beams

$x_1 \xrightarrow{<x> + \delta x_1} \rightarrow$  
$x_2 \xrightarrow{<x> + \delta x_2} \rightarrow$  
$\delta x_1 - \delta x_2$

For fluctuations (normalized to SNL) $G = 1$ for any classical beams.  
(regardless of the Fano factor F)

If $G < 1$ the beams (with equal mean intensities) have better-than-classical correlated fluctuations  
$=> \text{“twin beams”}$
balanced non-classical beams

\[ G < 1 \text{ implies that } C_{12} > 1 - \frac{1}{F} \]

This applies to “intensity twin beams” as well as to any “quadrature twin beams” that are formed by mixing a squeezed vacuum state and a coherent state on a 50/50 beamsplitter.

\[ F_i = \langle \delta X_i^2 \rangle \]
unbalanced case

use a non-50/50 beamsplitter

\[
|C_{12}|_{\text{classical}} = \sqrt{(1 - \frac{1}{F_1})(1 - \frac{1}{F_2})}
\]

this leads to \( G \geq 1 \)

If \( F_1 = 1 \) or if \( F_2 = 1 \) then \( C_{12} > 0 \) leads to \( G < 1 \)

(and the above formula doesn’t apply!)

\( F = 1 \implies \text{shot noise} \)

Thus, correlations with shot noise are a quantum effect!
An operational definition of gemelity is: The minimum noise (normalized to shot noise) obtained when mixing two beams on a variable-transmission beamsplitter.

In the lab one almost never has exactly equal-intensity beams to look at, so this is a useful idea for unequal beams.
real twin beams

“twin beams” from 4WM are twin only in some approximation
- intensity-difference squeezing
- phase-sum squeezing

- our “twin beams” are not even of the same color light!
  - we get the same number of photons in each beam
  - have a phase relation - with two different frequencies the phases in the two beams run at different rates, but:
    \[ \phi_1 + \phi_2 = 2\phi_0 \quad \text{or} \quad \phi_{\text{probe}} + \phi_{\text{conjugate}} = 2\phi_{\text{pump}} \]
quantum non-demolition

If two observables $M_1$ and $M_2$ are perfectly correlated, then measuring one obviously tells you the value of the other without having to measure it.

-> Measuring $M_2$ is a “quantum non-demolition measurement” of $M_1$.

This is only really interesting if you can measure quantum properties, of course.

-> If the measurements are sufficiently precise that one can determine the quantum fluctuations then one can measure $M_2$ and “feed forward” to $M_1$ to produce a non-classical field.
Beyond gemelity this is another level of quantum correlation

QND if: \( V(X_1 | X_2) < 1 \)

that is, the conditional variance of \( X_1 \), given a measured value of \( X_2 \), is sub-shot-noise

\[
V_{1|2} \equiv V(X_1 | X_2) = F_1(1 - C_{12}^2)
\]
balanced case QND

balanced case - same mean intensity and noise variance in each beam

\[ V_{1|2} = V_{2|1} = V < 1 \]

\[ V = G(1 + C_{12}) = 2G - \frac{G^2}{F} \]

\[ G \leq V \leq 2G \]

⇒ All QND correlated beams are “twin beams” but not all twin beams have QND levels of correlations
unbalanced QND

If the conditional variances are not equal, then there are two QND criteria

\[ V_{1|2} < 1 \quad \text{and} \quad V_{2|1} < 1 \]

a QND measurement in this case is not symmetric

if \( V_{1|2} < 1 \) or \( C_{12} > \sqrt{1 - \frac{1}{F_1}} \)

then one can make a QND measurement on beam 1 using the correlation between beams 1 and 2.
inseparability

=> A system that can only be described by an entangled or non-separable state.
This is a stronger criteria for “quantumness” than either the QND or gemelity (“twinness”) conditions.

For a pure state, if $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \sim$ tensor product

Operators in the separate spaces will yield a mean value that is a simple product of the individual means. They are uncorrelated.

$\Rightarrow$ Any correlation between observables implies entanglement for pure states!
(it does not tell you the level of entanglement or “quantumness”!)
inseparability for mixtures

For a statistical mixture
- density matrix description
- now correlations do NOT imply entanglement

Correlations can result from separable states, that is, classical statistical mixtures of factorizable states

$$\rho = \sum_j p_j \psi_1 j > \otimes \psi_2 j < \psi_1 j \otimes \psi_2 j$$

$$\sum_j p_j = 1$$
\[ \rho = \sum_n p_n |n,n><n,n| \]

-> Has equal photon number \( n \) in each of two modes
   (such as generated by PDC or 4WM)

thus \( C_{12} = 1, G = 0 \) and \( V_{1|2} = V_{2|1} = 0 \)

-> photon number states are indeed “quantum” but this state is neither entangled nor is it non-separable
inseparability

-> For this measure of “quantumness” you need to measure two joint correlations on non-commuting observables

\[
X_- = \frac{1}{\sqrt{2}}(X_1 - X_2) \quad \text{and} \quad Y_+ = \frac{1}{\sqrt{2}}(Y_1 + Y_2)
\]

“Duan criterion” \( I = \text{var}(X_-) + \text{var}(Y_+) = \langle \Delta X_-^2 \rangle + \langle \Delta Y_+^2 \rangle \) (variances normalized to SNL)

\( I < 2 \Rightarrow \text{inseparable} \)
inseparability

-> In the lab we often do not have exactly balanced signals, so that we use a variable gain $g$ in the relation

$$X_\pm = \frac{1}{\sqrt{2}} (X_1 - gX_2) \quad \text{and} \quad Y_\pm = \frac{1}{\sqrt{2}} (Y_1 + gY_2)$$

"Duan criterion" $I = \text{var}(X_-) + \text{var}(Y_+) = \langle \Delta X_-^2 \rangle + \langle \Delta Y_+^2 \rangle$

$I < 2 \implies$ inseparable

This implies quantum correlations from an entangled or non-separable state.
implications of inseparability

Classical beams will give $G > 1$ for any measured variables. This implies that $I > 2$; the beams are separable.

On the other hand, if $I < 2$, then $G < 1$ for at least one of the two beams and the beams are “twin beams” in either intensity or in phase.

$$F_i = \langle \delta X_i^2 \rangle \quad G = \frac{\langle (\delta X_1 - \delta X_2)^2 \rangle}{2} \quad (\text{for equal means and noise}) \quad I = \langle \Delta X_-^2 \rangle + \langle \Delta Y_+^2 \rangle$$
In their 1935 paper Einstein, Podolsky, and Rosen wrote down a wavefunction where the positions of two particles were perfectly correlated and the momenta are perfectly anti-correlated.

\[
\psi(x_1, x_2) = \int_{-\infty}^{\infty} e^{ip(x_1 - x_2 + x_0)2\pi/h} \, dp \quad x_0 = \text{const.}
\]

This state apparently allows one to make perfect QND measurements on \(x_1\) and \(p_1\) by measuring \(x_2\) and \(p_2\). This would violate the Heisenberg uncertainty relations if it were true.
EPR

This violation of the uncertainty relations is only “apparent,” as one can only measure $x_2$ or $p_2$ - you get to choose, but you cannot measure both simultaneously!

The EPR inequality*

$$E_{12} = \text{var}(X_1|X_2) \text{var}(Y_1|Y_2) = \mathcal{V}_{X_1|X_2} \mathcal{V}_{Y_1|Y_2} < 1.$$  

discriminates this additional level of “quantumness.”

This product of conditional variances is a stronger condition than inseparability; all EPR states are inseparable, while all inseparable states do not violate the EPR inequality.

demonstrating entanglement

alignment and bright beam entanglement

scan LO phase

phase stable local oscillators at +/- 3GHz from the pump
Seeded 4-wave mixing

seeded, bright modes

conjugate

probe seed

$^85\text{Rb}$

mask

pump

probe

$\Delta \theta$

cone of vacuum-squeezed modes
(allowed by phase matching)
Local Oscillators

- Local oscillators are generated with a second 4MW process:
  - Phase-locked LOs travel together with signal using same optics until near detector
  - Generate required frequencies
  - Automatic phase-matching
  - Can use arbitrary shape

- The LO selects the “shape” of the beam that is measured by the homodyne detection.

- At this level, excess noise of LOs has no effect for properly balanced homodyne detection.
demonstrating entanglement

vacuum squeezing

unsqueezed vacuum

probe

conjugate

pumps

50/50 BS

signal pump

LO pump

+ and -

scan LO phase

pzt mirror

pzt mirror
EPR violation:
$E_{12} = \text{Var}(X_1|X_2)\text{var}(Y_1|Y_2) < 1$

violation guaranteed with 3dB squeezing in each quadrature

measurements at 0.5 MHz
Bell violations

The Bell inequality and its violation tells one that the correlations in a system cannot be described by a local hidden variable theory. This is yet another level of quantum correlation beyond that of EPR beams. This has fascinating philosophical, as well as experimental physical implications...

The twin-beam states generated by 4WM are all Gaussian states. Thus, there will always be a quasi-probability distribution that can be used to construct a local hidden variable theory for these states that will “explain” or account for the correlations between the beams.

Violation of a Bell inequality requires a non-Gaussian state, or a non-Gaussian measurement operation.
Gaussian states

phase space distribution functions have Gaussian profiles

coherent states have symmetric distributions

squeezed states have asymmetric distributions

non-Gaussian states with negative-going Wigner functions that do not represent classical probability functions

figures from “A guide to Experiments in Quantum Optics” by H. Bachor and T. Ralph and “Quantum Optics” by G.S. Agarwal
single photon detection is a non-Gaussian operation

One can perform Bell tests based on the detection of single quanta and for many years Bell tests based on single photon detection (a non-Gaussian operation) have been refined.
Bell violations

Since our twin-beam states are all Gaussian states, we require a non-Gaussian measurement operation if we want to explore this level of “quantumness” using them...

“Photon subtraction” – that is, the absorption of a single photon from a field – is such a non-Gaussian operation.

top figures from “Quantum Optics” by G.S. Agarwal
homodyne detection is a Gaussian operation

The local oscillator is typically a coherent state; a Poisson distribution with large $\langle n \rangle$, and thus a nearly Gaussian distribution about the mean. Interfering it with a small signal will always result in a nearly-Gaussian distribution being detected.
Bell test with CVs

• “photon-subtracted” states allow for a negative-going Wigner function after the beamsplitter and a Bell inequality violation

homodyne detection of “photon-subtracted” states

- “discretize” the CV quadrature measurement to binary +/- 1
- maximal violation is 2.05 or about 2%
- pretty much independent of PD efficiency but need ~90% homodyne efficiency; optimizes with about 6 dB squeezing
general proposal (Garcia-Patron)

we have already shown CV EPR violations; but not loophole-free... and nothing outside the light cone

photon-subtracted states required for these proposals have been generated conditionally (Grangier; NIST Boulder; single mode, but 2-mode versions are a clear extension)

-discretize CV spectra to +/-1 for positive/zero and negative results
-use CHSH inequality for two possible measurements $A_1$ and $A_2$ at $A$, and $B_1$ and $B_2$ at $B$ with results $a_1$, $a_2$, $b_1$ and $b_2$

$$S = <a_1 b_1> + <a_1 b_2> + <a_2 b_1> - <a_2 b_2>$$

for local-realistic models $|S| \leq 2$

-can somewhat compensate efficiency and detector noise with increased squeezing
loophole-free tests

Several existing “loophole-free” tests have been recently published based on discrete variable violations:

Two using polarization states of individual photons:

One using electron spin states in diamond: